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ON THE CLASS OF CAUSTICS BY REFLECTION OF PLANAR CURVES

ALFREDERIC JOSSE AND FRANÇOISE PÈNE

ABSTRACT. Given any light position $S \in \mathbb{P}^2$ and any algebraic curve \mathcal{C} of \mathbb{P}^2 (with any kind of singularities), we consider the incident lines coming from S (i.e. the lines containing S) and their reflected lines after reflection on the mirror curve \mathcal{C} . The caustic by reflection $\Sigma_S(\mathcal{C})$ is the Zariski closure of the envelope of these reflected lines. We introduce the notion of reflected polar curve and express the class of $\Sigma_S(\mathcal{C})$ in terms of intersection numbers of \mathcal{C} with the reflected polar curve, thanks to a fundamental lemma established in [15]. This approach enables us to state an explicit formula for the class of $\Sigma_S(\mathcal{C})$ in every case in terms of intersection numbers of the initial curve \mathcal{C} .

INTRODUCTION

Let \mathbf{V} be a three dimensional complex vector space endowed with some fixed basis. We consider a light point $\mathcal{S}[x_0 : y_0 : z_0] \in \mathbb{P}^2 := \mathbb{P}(\mathbf{V})$ and a mirror given by an irreducible algebraic curve $\mathcal{C} = V(F)$ of \mathbb{P}^2 , with $F \in \text{Sym}^d(\mathbf{V}^\vee)$ (F corresponds to a polynomial of degree d in $\mathbb{C}[x, y, z]$). We denote by d^\vee the class of \mathcal{C} . We consider the caustic by reflection $\Sigma_S(\mathcal{C})$ of the mirror curve \mathcal{C} with source point S . Recall that $\Sigma_S(\mathcal{C})$ is the Zariski closure of the envelope of the reflected lines associated to the incident lines coming from S after reflection off \mathcal{C} . When S is not at infinity, Quetelet and Dandelin [17, 9] proved that the caustic by reflection $\Sigma_S(\mathcal{C})$ is the evolute of the S -centered homothety (with ratio 2) of the pedal of \mathcal{C} from S (i.e. the evolute of the orthotomic of \mathcal{C} with respect to S). This decomposition has also been used in a modern approach by [2, 3, 4] to study the source genericity (in the real case). In [15] we stated formulas for the degree of the caustic by reflection of planar algebraic curves.

In [7], Chasles proved that the class of $\Sigma_S(\mathcal{C})$ is equal to $2d^\vee + d$ for a generic (\mathcal{C}, S) . In [1], Brocard and Lemoyne gave (without any proof) a more general formula only when S is not at infinity. The Brocard and Lemoyne formula appears to be the direct composition of formulas got by Salmon and Cayley in [18, p. 137, 154] for some geometric characteristics of evolute and pedal curves. The formula given by Brocard and Lemoyne is not satisfactory for the following reasons. The results of Salmon and Cayley apply only to curves having no singularities other than ordinary nodes and cusps [18, p. 82], but the pedal of such a curve is not necessarily a curve satisfying the same properties. For example, the pedal curve of the rational cubic $V(y^2z - x^3)$ from $[4 : 0 : 1]$ is a quartic curve with a triple ordinary point. Therefore it is not correct to compose directly the formulas got by Salmon and Cayley as Brocard and Lemoyne apparently did (see also Section 5 for a counterexample of the Brocard and Lemoyne formula for the class of the caustic by reflection).

Let us mention some works on the evolute and on its generalization in higher dimension [10, 19, 6]. In [10], Fantechi gave a necessary and sufficient condition for the birationality of the evolute of a curve and studied the number and type of the singularities of the general evolute. Let

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us insist on the fact that there exist irreducible algebraic curves (other than lines and circles) for which the evolute map is not birational. This study of evolute is generalized in higher dimension by Trifogli in [19] and by Catanese and Trifogli [6].

The aim of the present paper is to give a formula for the class (with multiplicity) of the caustic by reflection for any algebraic curve \mathcal{C} (**without any restriction neither on the singularity points nor on the flex points**) and **for any light position S** (including the case when S is at infinity or when S is on the curve \mathcal{C}).

In Section 1, we define the reflected lines \mathcal{R}_m at a generic $m \in \mathcal{C}$ and the (rational) “reflected map” $R_{\mathcal{C},S} : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ mapping a generic $m \in \mathcal{C}$ to the equation of \mathcal{R}_m .

In section 2, we define the caustic by reflection $\Sigma_S(\mathcal{C})$, we give conditions ensuring that $\Sigma_S(\mathcal{C})$ is an irreducible curve and we prove that its class is the degree of the image of \mathcal{C} by $R_{\mathcal{C},S}$.

In section 3, we give formulas for the class of caustics by reflection valid for any (\mathcal{C}, S) . These formulas describe precisely how the class of the caustic depends on geometric invariants of \mathcal{C} and also on the relative positions of S and of the two cyclic points I, J with respect to \mathcal{C} . As a consequence of this result, we obtain the following formula for the class of $\Sigma_S(\mathcal{C})$ valid for any \mathcal{C} of degree $d \geq 2$ and for a generic source position S :

$$\text{class}(\Sigma_S(\mathcal{C})) = 2d^\vee + d - \Omega(\mathcal{C}, \ell_\infty) - \mu_I(\mathcal{C}) - \mu_J(\mathcal{C}),$$

where $\Omega(\mathcal{C}, \ell_\infty)$ is the contact number of \mathcal{C} with the line at infinity ℓ_∞ and with $\mu_I(\mathcal{C})$ and $\mu_J(\mathcal{C})$ are the multiplicities number of respectively I and J on \mathcal{C} .

In Section 4, our formulas are illustrated on two examples of curves (the lemniscate of Bernoulli and the quintic considered in [15]).

In section 5, we compare our formula with the one given by Brocard and Lemoyne for a light position not at infinity. We also give an explicit counter-example to their formula.

In Section 6, we prove our main theorem. In a first time, we give a formula for the class of the caustic in terms of intersection numbers of \mathcal{C} with a generic “reflected polar” at the base points of $R_{\mathcal{C},S}$. In a second time, we compute these intersection numbers in terms of the degree d and of the class d^\vee of \mathcal{C} but also in terms of intersection numbers of \mathcal{C} with each line of the triangle IJS .

In appendix A, we prove a useful formula expressing the classical intersection number in terms of probranches.

1. REFLECTED LINES \mathcal{R}_m AND RATIONAL MAP $R_{\mathcal{C},S}$

Recall that we consider a light position $S[x_0 : y_0 : z_0] \in \mathbb{P}^2$ and an irreducible algebraic (mirror) curve $\mathcal{C} = V(F)$ of \mathbb{P}^2 given by a homogeneous polynomial $F \in \text{Sym}^d(\mathbf{V})$ with $d \geq 2$. We write $\text{Sing}(\mathcal{C})$ for the set of singular points of \mathcal{C} . For any non singular point m , we write $\mathcal{T}_m\mathcal{C}$ for the tangent line to \mathcal{C} at m . We set $\mathbf{S}(x_0, y_0, z_0) \in \mathbf{V} \setminus \{\mathbf{0}\}$. For any $m[x : y : z] \in \mathbb{P}^2$, we write $\mathbf{m}(x, y, z) \in \mathbf{V} \setminus \{\mathbf{0}\}$. We write as usual $\ell_\infty = V(z) \subset \mathbb{P}^2$ for the line at infinity. For any $\mathbf{P}(x_1, y_1, z_1) \in \mathbf{V} \setminus \{\mathbf{0}\}$, we define

$$\Delta_{\mathbf{P}}F := x_1F_x + y_1F_y + z_1F_z \in \text{Sym}^{d-1}(\mathbf{V}^\vee).$$

Recall that $V(\Delta_{\mathbf{P}}F)$ is the polar curve of \mathcal{C} with respect to $P[x_1 : y_1 : z_1] \in \mathbb{P}^2$.

Since the initial problem is euclidean, we endow \mathbb{P}^2 with an angular structure for which $I[1 : i : 0] \in \mathbb{P}^2$ and $J[1 : -i : 0] \in \mathbb{P}^2$ play a particular role. To this end, let us recall the definition of the cross-ratio β of 4 points of ℓ_∞ . Given four points $(P_i[a_i : b_i : 0])_{i=1,\dots,4}$ such

that each point appears at most 2 times, we define the cross-ratio $\beta(P_1, P_2, P_3, P_4)$ of these four points as follows :

$$\beta(P_1, P_2, P_3, P_4) = \frac{(b_3a_1 - b_1a_3)(b_4a_2 - b_2a_4)}{(b_3a_2 - b_2a_3)(b_4a_1 - b_1a_4)},$$

with convention $\frac{1}{0} = \infty$. For any distinct lines \mathcal{A}_1 and \mathcal{A}_2 not equal to ℓ_∞ , containing neither I nor J , we define the oriented angular measure between \mathcal{A}_1 and \mathcal{A}_2 by θ (modulo $\pi\mathbb{Z}$) such that

$$e^{-2i\theta} = \beta(P_1, P_2, I, J) = \frac{(a_1 + ib_1)(a_2 - ib_2)}{(a_1 - ib_1)(a_2 + ib_2)}$$

(where $P_i[a_i : b_i : 0]$ is the point at infinity of \mathcal{A}_i). Let $Q \in \text{Sym}^2(\mathbf{V}^\vee)$ be defined by $Q(x, y) := x^2 + y^2$. It will be worth noting that $Q(\nabla F) = F_x^2 + F_y^2 = \Delta_I F \Delta_J F$. For every non singular point m of $\mathcal{C} \setminus \ell_\infty$, we recall that $t_m[F_y : -F_x : 0] \in \mathbb{P}^2$ is the point at infinity of $\mathcal{T}_m \mathcal{C}$ and so $t_m \notin \{I, J\}$ is equivalent to $m \notin V(Q(\nabla F))$.

Now, for any $m \in \mathcal{C} \setminus (\ell_\infty \cup Q(\nabla F))$ and any incident line ℓ containing m , we define as follows the associated reflected line $\mathfrak{R}_m(\ell)$ (for the reflexion on \mathcal{C} at m with respect to the Snell-Descartes reflection law $\text{Angle}(\ell, \mathcal{T}_m) = \text{Angle}(\mathcal{T}_m, \mathfrak{R}_m)$).

Definition 1. For every $m \in \mathcal{C} \setminus (\ell_\infty \cup V(Q(\nabla F)))$, we define $\mathfrak{r}_m : \ell_\infty \rightarrow \ell_\infty$ mapping $P \in \ell_\infty$ to the unique $\mathfrak{r}_m(P)$ such that $\beta(P, t_m, I, J) = \beta(t_m, \mathfrak{r}_m(P), I, J)$.

We define $\mathfrak{R}_m : \mathcal{F}_m \rightarrow \mathcal{F}_m$ with $\mathcal{F}_m := \{\ell \in G(1, \mathbb{P}^2), m \in \ell\}$ by $\mathfrak{R}_m(\ell) = (m \mathfrak{r}_m(P_\ell))$ if P_ℓ is the point at infinity of ℓ .

We have (on coordinates)

$$\mathfrak{r}_m([x_1 : y_1 : 0]) = [x_1(F_x^2 - F_y^2) + 2y_1F_xF_y : -y_1(F_x^2 - F_y^2) + 2x_1F_xF_y : 0]$$

Remark 2. Observe that \mathfrak{r}_m is an involution on $\ell_\infty \cong \mathbb{P}^1$ with exactly two fixed points t_m and $n_m[F_x : F_y : 0]$. As a consequence, \mathfrak{R}_m is an involution with two fixed points $\mathcal{T}_m(\mathcal{C})$ and $\mathcal{N}_m(\mathcal{C}) := (mn_m)$ the normal line to \mathcal{C} at m .

Moreover $\mathfrak{r}_m(I) = J$ and $\mathfrak{r}_m(J) = I$.

Definition 3. For any $m[x : y : z] \in \mathcal{C} \setminus (\{S\} \cup \ell_\infty \cup V(Q(\nabla F)))$ we define the **reflected line** \mathcal{R}_m on \mathcal{C} at m (of the incident line coming from S) as the line $\mathcal{R}_m := \mathfrak{R}_m((mS))$.

For $m[x : y : z] \in \mathcal{C} \setminus (\{S\} \cup \ell_\infty \cup V(Q(\nabla F)))$, the point at infinity of (Sm) is $s_m[x_0z - z_0x : y_0z - z_0y : 0]$. Due to the Euler identity, on \mathcal{C} , we have $xF_x + yF_y + zF_z = 0$ and so $(x_0z - z_0x)F_x + (y_0z - z_0y)F_y = z\Delta_S F$. Hence $\mathfrak{r}(s_m) = [-v_m : u_m : 0]$ and the reflected line \mathcal{R}_m is the set of $P[X : Y : Z] \in \mathbb{P}^2$ such that $u_m X + v_m Y + w_m Z = 0$, with

$$\begin{aligned} u_m &:= (z_0y - zy_0)(F_x^2 + F_y^2) + 2z\Delta_S F.F_y \in \text{Sym}^{2d-1}(\mathbf{V}^\vee) \\ v_m &:= (zx_0 - z_0x)(F_x^2 + F_y^2) - 2z\Delta_S F.F_x \in \text{Sym}^{2d-1}(\mathbf{V}^\vee) \\ w_m &:= \frac{-xu_m - yv_m}{z} = (xy_0 - yx_0)(F_x^2 + F_y^2) - 2\Delta_S F(xF_y - yF_x) \in \text{Sym}^{2d-1}(\mathbf{V}^\vee). \end{aligned}$$

Definition 4. We call **reflected map of \mathcal{C} from S** the following rational map

$$R_{\mathcal{C}, S} : \begin{array}{ccc} \mathbb{P}^2 & \rightarrow & \mathbb{P}^2 \\ m & \mapsto & [u_m : v_m : w_m] \end{array}.$$

We also define the rational map $T_{\mathcal{C}, S} := (R_{\mathcal{C}, S})|_{\mathcal{C}} : \mathcal{C} \rightarrow \mathbb{P}^2$.

For any $\mathbf{m} \in \mathbf{V}$, it will be useful to define $\mathbf{R}_{F,S}(\mathbf{m}) := (u_{\mathbf{m}}, v_{\mathbf{m}}, w_{\mathbf{m}}) \in \mathbf{V}$ and to notice that

$$\mathbf{R}_{F,S}(\mathbf{m}) = Q(\nabla F(\mathbf{m})) \cdot (\mathbf{m} \wedge \mathbf{S}) - 2\Delta_S F(\mathbf{m}) \cdot (\mathbf{m} \wedge \mathbf{n}_{\mathbf{m}}) \in \mathbf{V},$$

with¹ $\mathbf{n}_{\mathbf{m}}(F_x(\mathbf{m}), F_y(\mathbf{m}), 0) \in \mathbf{V}$.

Proposition 5. *The base points of $T_{\mathcal{C},S}$ are the following:*

I, J, S (if these points are in \mathcal{C}), the singular points of \mathcal{C} and the points of tangency of \mathcal{C} with some line of the triangle (IJS) .

Proof. We have to prove that the set of base points of $T_{\mathcal{C},S}$ is the following set: $\mathcal{M} := \mathcal{C} \cap (\{I, J, S\} \cup V(\Delta_S F, Q(\nabla F)) \cup V(F_x, F_y))$. We just prove that $\text{Base}(T_{\mathcal{C},S}) \subset \mathcal{M}$, the converse being obvious (observe that if $m \in \{I, J\}$, we automatically have $Q(\nabla F(\mathbf{m})) = 0$ and $\mathbf{n}_{\mathbf{m}} \in \text{Vect}(\mathbf{m})$). Let $m[x; y; z] \in \mathcal{C}$ be such that $\mathbf{R}_{F,S}(\mathbf{m}) = \mathbf{0}$. Then \mathbf{m} and $Q(\nabla F(\mathbf{m})) \cdot \mathbf{S} - 2\Delta_S F(\mathbf{m}) \cdot \mathbf{n}_{\mathbf{m}}$ are colinear. Due to the Euler identity, we have $0 = DF(\mathbf{m}) \cdot \mathbf{m}$ (with $DF(\mathbf{m})$ the differential of F at \mathbf{m}) and so $0 = -\Delta_S F(\mathbf{m}) \cdot Q(\nabla F(\mathbf{m}))$ since $DF(\mathbf{m}) \cdot \mathbf{S} = \Delta_S F(\mathbf{m})$ and since $DF(\mathbf{m}) \cdot \mathbf{n}_{\mathbf{m}} = Q(\nabla F(\mathbf{m}))$. Hence $\Delta_S F(\mathbf{m}) = 0$ or $Q(\nabla F(\mathbf{m})) = 0$.

If $\Delta_S F(\mathbf{m}) = 0$, then either $Q(\nabla F(\mathbf{m})) = 0$ or $m = S$.

If $\Delta_S F(\mathbf{m}) \neq 0$ and $Q(\nabla F(\mathbf{m})) = 0$, then $F_x(\mathbf{m}) = F_y(\mathbf{m}) = 0$ or $m = [F_x(\mathbf{m}) : F_y(\mathbf{m}) : 0]$. Assume that $m = [F_x(\mathbf{m}) : F_y(\mathbf{m}) : 0]$. Then, since $Q(\nabla F(\mathbf{m})) = 0$, we conclude that $m \in \{I, J\}$. \square

In the following result, we state the S -generic birationality of $T_{\mathcal{C},S}$. We give a short version of the proof of [16]. Let us indicate that another proof of the same result has been established at the same period by Catanese in [5].

Proposition 6 (see also [16, 5]). *Let \mathcal{C} be an irreducible curve of degree $d \geq 2$. Then, for a generic $S \in \mathbb{P}^3$, the map $T_{\mathcal{C},S}$ is birational.*

Proof. For every $m \in \mathcal{C}_0 := \mathcal{C} \setminus (\ell_\infty \cup V(Q(\nabla F)))$ and every $S \in \mathbb{P}^2 \setminus \{m\}$, we write $\mathcal{R}_{m,S}$ for the reflected line $\mathfrak{R}_m((mS))$. For every $m \in \mathcal{C}_0$, we consider the set $K_m := \{S \in \mathbb{P}^2 \setminus \mathcal{C} : \exists m' \in \mathcal{C}_0 \setminus \{m\}, \mathcal{R}_{m,S} = \mathcal{R}_{m',S}\}$.

- Let us prove that, for any $m \in \mathcal{C}_0$, K_m is contained in an algebraic curve \bar{K}_m of degree less than $2d^2 + 2$.

Let $m \in \mathcal{C}_0$. Consider $S \in \mathbb{P}^2 \setminus \mathcal{C}$ and $m' \in \mathcal{C}_0 \setminus \{m\}$ such that $\mathcal{R}_{m,S} = \mathcal{R}_{m',S}$. Then we have $\mathcal{R}_{m,S} = \mathcal{R}_{m',S} = (mm')$ and so $S \in \mathcal{R}_{m,m'} \cap \mathcal{R}_{m',m}$.

Assume first that $\mathcal{R}_{m,m'} = \mathcal{R}_{m',m}$. Then this line is (mm') and it is its own reflected line both at m and at m' . This implies that (mm') is either $\mathcal{T}_m \mathcal{C}$ or $\mathcal{N}_m \mathcal{C}$, so that $S \in \mathcal{T}_m \mathcal{C} \cup \mathcal{N}_m \mathcal{C}$.

Assume now that $\mathcal{R}_{m,m'} \neq \mathcal{R}_{m',m}$. Then $S = \tau_m(m')$ with $\tau_m : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ the rational map associated to $\tau_{\mathbf{m}} : \mathbf{V} \rightarrow \mathbf{V}$ with $\tau_{\mathbf{m}}(\mathbf{m}') = \mathbf{R}_{F,\mathbf{m}'}(\mathbf{m}) \wedge \mathbf{R}_{F,\mathbf{m}}(\mathbf{m}')$. Hence $K_m \subseteq \bar{K}_m := \mathcal{T}_m \mathcal{C} \cup \mathcal{N}_m \mathcal{C} \cup \overline{\tau_m(\mathcal{C})}$, where \bar{A} is the Zariski closure of a set A . Since the degree (in m') of the coordinates of $\tau_{\mathbf{m}}$ is $2d$, we conclude that $\deg \bar{K}_m \leq 2d^2 + 2$.

- The set K of points $S \in \mathbb{P}^2 \setminus \mathcal{C}$ such that $R_{\mathcal{C},S}$ is not birational is contained in

$$\bar{K} := \bigcup_{E \subset \mathcal{C}_0 : \#E < \infty} \bigcap_{m \in \mathcal{C}_0 \setminus E} \bar{K}_m.$$

¹with $\wedge : \mathbf{V} \times \mathbf{V} \rightarrow \mathbf{V}$ being given in coordinates by $(x_1, y_1, z_1) \wedge (x_2, y_2, z_2) = \begin{pmatrix} z_2 y_1 - z_1 y_2 \\ z_1 x_2 - z_2 x_1 \\ x_1 y_2 - y_1 x_2 \end{pmatrix}$.

To conclude we will apply the Zorn lemma. We have to prove that $\{\bigcap_{m \in \mathcal{C}_0 \setminus E} \bar{K}_m, \#E < \infty\}$ is inductive for the inclusion. Let $(\mathcal{F}_j := \bigcap_{m \in \mathcal{C}_0 \setminus E_j} \bar{K}_m)_{j \geq 1}$ be an increasing sequence of sets (with E_j finite subsets of \mathcal{C}_0). Write Z for the union of these sets. Observe that $Z \subseteq \bar{K}_{m_0}$ for some fixed $m_0 \in \mathcal{C}_0 \setminus \bigcup_{i \geq 1} E_i$. The set \bar{K}_{m_0} is the union of irreducible algebraic curves C_1, \dots, C_p . We write d_i for the degree of C_i . If $C_i \subseteq Z$, we write $N_i := \min\{j \geq 1 : C_i \subset \mathcal{F}_j\}$. If $C_i \not\subseteq Z$, then $(C_i \cap \mathcal{F}_j)_{j \geq 1}$ is an increasing sequence of finite sets containing at most $d_i(2d^2 + 2)$ points and we set $N_i := \min\{j : (C_i \cap Z) \subseteq \mathcal{F}_j\}$. We obtain $Z = \mathcal{F}_{\max(N_1, \dots, N_p)}$. Due to the Zorn lemma, there exists a finite set E_0 such that $K \subset \bigcap_{m \in \mathcal{C}_0 \setminus E_0} \bar{K}_m$, from which the result follows. \square

2. CAUSTIC BY REFLECTION

Definition 7. *The **caustic by reflection** $\Sigma_S(\mathcal{C})$ is the Zariski closure of the envelope of the reflected lines $\{\mathcal{R}_m; m \in \mathcal{C} \setminus (\{S\} \cup \ell_\infty \cup V(Q(\nabla F)))\}$.*

Recall that, in [15], we have defined a rational map $\Phi_{F,S}$ called **caustic map** mapping a generic $m \in \mathcal{C}$ to the point of tangency of $\Sigma_S(\mathcal{C})$ with \mathcal{R}_m and that $\Sigma_S(\mathcal{C})$ is the Zariski closure of $\Phi_{F,S}(\mathcal{C})$.

In the present work, we will not consider the cases in which the caustic by reflection $\Sigma_S(\mathcal{C})$ is a single point. We recall that these cases are easily characterized as follows.

Proposition 8. *Assume that*

- (i) $S \notin \{I, J\}$,
- (ii) \mathcal{C} is not a line (i.e. $d \neq 1$),
- (iii) if $d = 2$, then S is not a focus of the conic \mathcal{C} .

Then $\Sigma_S(\mathcal{C})$ is not reduced to a point and is an irreducible curve.

Proof. Assume (i), (ii) and (iii) and that $\Sigma_S(\mathcal{C}) = \{S'\}$ with $S' = [x_1 : y_1 : z_1]$.

When $S \notin \ell_\infty$, we will use the fact that $\Sigma_S(\mathcal{C})$ is the evolute of the orthotomic of \mathcal{C} with respect to S . Since \mathcal{C} is not a line, the orthotomic of \mathcal{C} with respect to S is not reduced to a point but its evolute is a point. This implies that the orthotomic of \mathcal{C} with respect to S is either a line (not equal to ℓ_∞) or a circle. But \mathcal{C} is the contrapedal (or orthocaustic) curve (from S) of the image by the S -centered homothety (with ratio $1/2$) of the orthotomic of \mathcal{C} . Therefore $d = 2$ and S is a focal point of \mathcal{C} , which contradicts (iii).

When $S \in \ell_\infty$ but $S' \notin \ell_\infty$, then, for symmetry reasons, we also have $\Sigma_{S'}(\mathcal{C}) = \{S\}$ and we conclude analogously.

Suppose now that $S, S' \in \ell_\infty$. We have $z_0 = z_1 = 0$. For every $m = [x : y : 1] \in \mathcal{C} \setminus (\ell_\infty \cup V(Q(\nabla F)))$, we have $\beta(S, t_m, I, J) = \beta(t_m, S', I, J)$. Therefore we have

$$\frac{(ix_0 - y_0)(-iF_y + F_x)}{(iF_y + F_x)(-ix_0 - y_0)} = \frac{(iF_y + F_x)(-ix_1 - y_1)}{(-iF_y + F_x)(ix_1 - y_1)}$$

and so

$$(ix_0 - y_0)(ix_1 - y_1)(-iF_y + F_x)^2 = (iF_y + F_x)^2(-ix_0 - y_0)(-ix_1 - y_1).$$

Now, according to (i), $ix_0 - y_0 \neq 0$, $-ix_0 - y_0 \neq 0$, $ix_1 - y_1 \neq 0$, $-ix_1 - y_1 \neq 0$. Hence $(-iF_y + F_x)^2 = a(iF_y + F_x)^2$ for some $a \neq 0$, which implies that $d = 1$ and contradicts (ii).

Hence we proved that $\Sigma_S(\mathcal{C})$ is not reduced to a point. Now the irreducibility of $\Sigma_S(\mathcal{C})$ comes from the fact that $\Sigma_S(\mathcal{C}) = \overline{\Phi_{F,\mathbf{S}}(\mathcal{C})}$ and that \mathcal{C} is an irreducible curve. \square

Proposition 9. *Assume that $\Sigma_S(\mathcal{C})$ is not reduced to a point. Then we have*

$$\text{class}(\Sigma_S(\mathcal{C})) = \deg(\overline{T_{\mathcal{C},S}(\mathcal{C})}), \quad (1)$$

where $\overline{T_{\mathcal{C},S}(\mathcal{C})}$ stands for the Zariski closure of $T_{\mathcal{C},S}(\mathcal{C})$.

Proof. This comes from the fact that $\Sigma_S(\mathcal{C})$ is the Zariski closure of the envelope of $\{\mathcal{R}_m, m \in \mathcal{C} \setminus (\text{Sing}(\mathcal{C}) \cup \{S\} \cup \ell_\infty \cup V(Q(\nabla F)))\}$ and can be precised as follows. For every algebraic curve $\Gamma = V(G)$ (with G in $\text{Sym}^k(\mathbf{V}^\vee)$ for some k), we consider the Gauss map $\delta_\Gamma : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ defined on coordinates by $\delta_\Gamma([x : y : z]) = [G_x : G_y : G_z]$, we obtain immediately the commutative diagram :

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{(\Phi_{F,\mathbf{S}})|_{\mathcal{C}}} & \Sigma_S(\mathcal{C}) \\ & \searrow T_{\mathcal{C},S} & \downarrow \delta_{\Sigma_S(\mathcal{C})} \\ & & \delta_{\Sigma_S(\mathcal{C})}(\Sigma_S(\mathcal{C})) \cong (\Sigma_S(\mathcal{C}))^\vee \end{array}, \quad (2)$$

with $\Phi_{F,\mathbf{S}}$ the caustic map defined in [15] (see the begining of the present section). \square

Let us notice that, according to the proof of Proposition 9, the rational map $T_{\mathcal{C},S}$ as the same degree as the rational map $(\Phi_{F,\mathbf{S}})|_{\mathcal{C}}$ (since $\Sigma_S(\mathcal{C})$ is irreducible and since the Gauss map $(\delta_{\Sigma_S(\mathcal{C})})|_{\Sigma_S(\mathcal{C})}$ is birational [11]).

3. FORMULAS FOR THE CLASS OF THE CAUSTIC

Since the map $T_{\mathcal{C},S}$ may be non birational, we introduce the notion of class with multiplicity of $\Sigma_S(\mathcal{C})$:

$$\text{mclass}(\Sigma_S(\mathcal{C})) = \delta_1(S, \mathcal{C}) \times \text{class}(\Sigma_S(\mathcal{C}))$$

where $\text{class}(\Sigma_S(\mathcal{C}))$ is the class of the algebraic curve $\Sigma_S(\mathcal{C})$ and where $\delta_1(S, \mathcal{C})$ is the degree of the rational map $T_{\mathcal{C},S}$. We recall that $\delta_1(S, \mathcal{C})$ corresponds to the number of preimages on \mathcal{C} of a generic point of $\Sigma_S(\mathcal{C})$ by $T_{\mathcal{C},S}$.

Before stating our main result, let us introduce some notations. For every $m_1 \in \mathbb{P}^2$, we write $\mu_{m_1} = \mu_{m_1}(\mathcal{C})$ for the multiplicity of m_1 on \mathcal{C} and consider the set $\text{Branch}_{m_1}(\mathcal{C})$ of branches of \mathcal{C} at m_1 . We denote by \mathcal{E} the set of couples point-branch (m_1, \mathcal{B}) of \mathcal{C} with $m_1 \in \mathcal{C}$ and $\mathcal{B} \in \text{Branch}_{m_1}(\mathcal{C})$. For every $(m_1, \mathcal{B}) \in \mathcal{E}$, we write $e_{\mathcal{B}}$ for the multiplicity of \mathcal{B} and $\mathcal{T}_{m_1}(\mathcal{B})$ the tangent line to \mathcal{B} at m_1 ; we observe that $\mu_{m_1} = \sum_{\mathcal{B} \in \text{Branch}_{m_1}(\mathcal{C})} e_{\mathcal{B}}$. We write $i_{m_1}(\Gamma, \Gamma')$ the intersection number of two curves Γ and Γ' at m_1 . For any algebraic curve \mathcal{C}' of \mathbb{P}^2 , we also define the contact number $\Omega_{m_1}(\mathcal{C}, \mathcal{C}')$ of \mathcal{C} and \mathcal{C}' at $m_1 \in \mathbb{P}^2$ by

$$\Omega_{m_1}(\mathcal{C}, \mathcal{C}') := i_{m_1}(\mathcal{C}, \mathcal{C}') - \mu_{m_1}(\mathcal{C})\mu_{m_1}(\mathcal{C}') \quad \text{if } m_1 \in \mathcal{C} \cap \mathcal{C}'$$

and

$$\Omega_{m_1}(\mathcal{C}, \mathcal{C}') := 0 \quad \text{if } m_1 \notin \mathcal{C} \cap \mathcal{C}'.$$

Recall that $\Omega_{m_1}(\mathcal{C}, \mathcal{C}') = 0$ means that $m_1 \notin \mathcal{C} \cap \mathcal{C}'$ or that \mathcal{C} and \mathcal{C}' intersect transversally at m_1 .

Theorem 10. *Assume that the hypotheses of Proposition 8 hold true.*

(1) *If $S \notin \ell_\infty$, the class (with multiplicity) of $\Sigma_S(\mathcal{C})$ is given by*

$$\text{mclass}(\Sigma_S(\mathcal{C})) = 2d^\vee + d - 2f' - g - f - g' + q', \quad (3)$$

where

- g is the contact number of \mathcal{C} with ℓ_∞ , i.e. $g := \sum_{m_1 \in \mathcal{C} \cap \ell_\infty} \Omega_{m_1}(\mathcal{C}, \ell_\infty)$,
- f is the multiplicity number at a cyclic point of \mathcal{C} with an isotropic line from S , i.e.

$$f := i_I(\mathcal{C}, (IS)) + i_J(\mathcal{C}, (JS)),$$

- f' is the contact number of \mathcal{C} with an isotropic line from S outside $\{I, J, S\}$, i.e.

$$f' := \sum_{m_1 \in (\mathcal{C} \cap (IS)) \setminus \{I, S\}} \Omega_{m_1}(\mathcal{C}, (IS)) + \sum_{m_1 \in (\mathcal{C} \cap (JS)) \setminus \{J, S\}} \Omega_{m_1}(\mathcal{C}, (JS)),$$

- g' given by $g' := i_S(\mathcal{C}, (IS)) + i_S(\mathcal{C}, (JS)) - \mu_S$;
- q' is given by

$$q' := \sum_{(m_1, \mathcal{B}) \in \mathcal{E}: m_1 \notin \{I, J, S\}, T_{m_1} \mathcal{B} = (IS) \text{ or } T_{m_1} \mathcal{B} = (JS), i_{m_1}(\mathcal{B}, T_{m_1}(\mathcal{B})) \geq 2e_{\mathcal{B}}} [i_{m_1}(\mathcal{B}, T_{m_1}(\mathcal{B})) - 2e_{\mathcal{B}}].$$

(2) If $S \in \ell_\infty$, the class of $\Sigma_S(\mathcal{C})$ is

$$mclass(\Sigma_S(\mathcal{C})) = 2d^\vee + d - 2g - \mu_I - \mu_J - \mu_S - c'(S), \quad (4)$$

with

$$c'(S) := \sum_{\mathcal{B} \in \text{Branch}_S(\mathcal{C}): i_S(\mathcal{B}, \ell_\infty) = 2e_{\mathcal{B}}} (e_{\mathcal{B}} + \min(i_S(\mathcal{B}, \text{Osc}_S(\mathcal{B})) - 3e_{\mathcal{B}}, 0)),$$

where $\text{Osc}_S(\mathcal{B})$ is any smooth algebraic osculating curve to \mathcal{B} at S (i.e. any smooth algebraic curve \mathcal{C}' such that $i_S(\mathcal{B}, \mathcal{C}') > 2e_{\mathcal{B}}$).

The notations introduced in this theorem are directly inspired by those of Salmon and Cayley [18] (see Section 5). Let us point out that, in this article, g is not the geometric genus of the curve.

Remark 11. Observe that we also have

$$c'(S) := \sum_{\mathcal{B} \in \text{Branch}_S(\mathcal{C}): i_S(\mathcal{B}, \ell_\infty) = 2e_{\mathcal{B}}} (e_{\mathcal{B}} + \min(\beta_1(S, \mathcal{B}) - 3e_{\mathcal{B}}, 0)),$$

where $\beta_1(S, \mathcal{B})$ is the first characteristic exponent of \mathcal{B} non multiple of $e_{\mathcal{B}}$ (see [22]).

Observe that, when $i_S(\mathcal{B}, T_S(\mathcal{B})) = 2e_{\mathcal{B}}$, we have $\min(i_S(\mathcal{B}, \text{Osc}_S(\mathcal{B})) - 3e_{\mathcal{B}}, 0) = 0$ except if S is a singular point and if the probranches of \mathcal{B} are given by $Y - x_0^{-1}y_0 = \alpha Z^2 + \alpha_1 Z^{\beta_1} + \dots$ in the chart $X = 1$ if $x_0 \neq 0$ (or $X - y_0^{-1}x_0 = \alpha Z^2 + \alpha_1 Z^{\beta_1} + \dots$ in the chart $Y = 1$ otherwise), with $\alpha \neq 0$, $\alpha_1 \neq 0$ and $2 < \beta_1 < 3$. Hence $c'(S) = \sum_{\mathcal{B} \in \text{Branch}_S(\mathcal{C}): i_S(\mathcal{B}, \ell_\infty) = 2e_{\mathcal{B}}} e_{\mathcal{B}}$ when \mathcal{C} admits no such branch tangent at S to ℓ_∞ .

Combining Proposition 6 and Theorem 10, we obtain

Corollary 12 (A source-generic formula for the class). *Let $\mathcal{C} \subset \mathbb{P}^2$ be a fixed curve of degree $d \geq 2$. For a generic source point S , we have $\delta_1(S, \mathcal{C}) = 1$ and $class(\Sigma_S(\mathcal{C})) = 2d^\vee + d - g - \mu_I - \mu_J$ with g the contact number of \mathcal{C} with ℓ_∞ .*

Proof. Due to Proposition 6, $\delta_1(S, \mathcal{C}) = 1$ for a generic $S \in \mathbb{P}^2$. So $class(\Sigma_S(\mathcal{C})) = mclass(\Sigma_S(\mathcal{C}))$.

Assume moreover, that $S \notin \ell_\infty$ (so we apply the first formula of Theorem 10), $S \notin \mathcal{C}$ (so $g' = 0$), that (IS) and (JS) are not tangent to \mathcal{C} (so $f' = q' = 0$ and $f = \mu_I(\mathcal{C}) + \mu_J(\mathcal{C})$). We obtain the result. \square

4. EXAMPLES

Let us now illustrate our result for two particular mirror curves.

4.1. Example of the lemniscate of Bernoulli. We consider the case when $\mathcal{C} = V(F)$ is the lemniscate of Bernoulli given by $F(x, y, z) = (x^2 + y^2)^2 - 2(x^2 - y^2)z^2$ and when $S \in \mathbb{P}^2 \setminus \{I, J\}$. The degree of \mathcal{C} is $d = 4$. The singular points of \mathcal{C} are $I[1 : i : 0]$, $J[1 : -i : 0]$ and $\mathcal{O}[0 : 0 : 1]$. These three points are double points, each one having two different tangent lines. Hence the class of \mathcal{C} is given by $d^\vee = d(d-1) - 3 \times 2 = 6$ and so

$$2d^\vee + d = 16.$$

The tangent lines to \mathcal{C} at I are $\ell_{1,I} := V(y - iz - ix)$ and $\ell_{2,I} := V(y - iz + ix)$ (the intersection number of \mathcal{C} with $\ell_{1,I}$ or with $\ell_{2,I}$ at I is equal to 4). The tangent lines to \mathcal{C} at J are $\ell_{1,J} := V(y + iz - ix)$ and $\ell_{2,J} := V(y + iz + ix)$ (the intersection number of \mathcal{C} with $\ell_{1,J}$ or with $\ell_{2,J}$ at J is equal to 4). This ensures that we have

$$f = 2(2 + \mathbf{1}_{S \in \ell_{1,I}} + \mathbf{1}_{S \in \ell_{2,I}} + \mathbf{1}_{S \in \ell_{1,J}} + \mathbf{1}_{S \in \ell_{2,J}}).$$

Observe that ℓ_∞ is not tangent to \mathcal{C} . Indeed I and J are the only points in $\mathcal{C} \cap \ell_\infty$ and ℓ_∞ is not tangent to \mathcal{C} at these points. Therefore we have $g = 0$ and $c'(S) = 0$.

Since I and J are also the only points at which \mathcal{C} is tangent to an isotropic line (i.e. a line containing I or J), we have $f' = 0$, $g' = \mu_S$, $q' = 0$. In this case, one can check that $\delta_1(S, \mathcal{C}) = 1$. Finally, we get

$$\text{if } S \notin \ell_\infty, \quad \text{class}(\Sigma_S(\mathcal{C})) = 12 - 2(\mathbf{1}_{S \in \ell_{1,I} \cup \ell_{2,I}} + \mathbf{1}_{S \in \ell_{1,J} \cup \ell_{2,J}}) - \mu_S. \quad (5)$$

Moreover, since $\mu_I = \mu_J = 2$, we have

$$\text{if } S \in \ell_\infty \setminus \{I, J\}, \quad \text{class}(\Sigma_S(\mathcal{C})) = 16 - 2 - 2 = 12, \quad (6)$$

(since $\mu_I = \mu_J = 2$ and since $\mu_S = 0$). For example, for $S[1 : 0 : 1]$, we get $\text{class}(\Sigma_S(\mathcal{C})) = 8$, since S is in $\ell_{2,I} \cap \ell_{1,J}$ but not in \mathcal{C} (so $\mu_S = 0$).

4.2. Example of a quintic curve. As in [15], we consider the quintic curve $\mathcal{C} = V(F)$ with $F(x, y, z) = y^2 z^3 - x^5$. We also consider a light point $S[x_0 : y_0 : z_0] \in \mathbb{P}^2 \setminus \{I, J\}$. This curve admits two singular points: $A_1[0 : 0 : 1]$ and $A_2[0 : 1 : 0]$, we have $d = 5$.

We recall that \mathcal{C} admits a single branch at A_1 , which has multiplicity 2 and which is tangent to $V(y)$. We observe that $i_{A_1}(\mathcal{C}, V(y)) = 5$.

Analogously, \mathcal{C} admits a single branch at A_2 , which has multiplicity 3 and which is tangent to ℓ_∞ . We observe that $i_{A_2}(\mathcal{C}, \ell_\infty) = 5$.

We obtain that the class of \mathcal{C} is $d^\vee = 5$ and that \mathcal{C} has no inflexion point (these two facts are proved in [15]). In particular, we get that $2d^\vee + d = 15$.

Since A_2 is the only point of $\mathcal{C} \cap \ell_\infty$, we get that $g = \Omega_{A_2}(\mathcal{C}, \ell_\infty) = 2$ and $f = 0$.

The curve \mathcal{C} admits six (pairwise distinct) isotropic tangent lines other than ℓ_∞ : ℓ_1, ℓ_2 and ℓ_3 containing I

$$\forall k \in \{1, 2, 3\}, \quad \ell_k = V\left(ix - y + \frac{3i}{25}\alpha^k \sqrt[3]{20}z\right), \quad \text{with } \alpha := e^{\frac{2i\pi}{3}}$$

and ℓ_4, ℓ_5 and ℓ_6 containing J :

$$\forall k \in \{1, 2, 3\}, \quad \ell_{3+k} = V\left(ix + y + \frac{3i}{25}\alpha^k \sqrt[3]{20}z\right).$$

For every $i \in \{1, 2, 3, 4, 5, 6\}$, we write a_i the point at which \mathcal{C} is tangent to ℓ_i (the points a_i correspond to the points of $\mathcal{C} \cap V(F_x^2 + F_y^2) \setminus \{A_1, A_2\}$). Since \mathcal{C} contains no inflexion point and since A_1 and A_2 are the only singular points of \mathcal{C} , we get that,

$$f' = \#\{i \in \{1, 2, 3, 4, 5, 6\} : S \in \ell_i \setminus \{a_i\}\} \quad \text{and} \quad q' = 0$$

when $S \notin \ell_\infty$.

Now recall that $g' = i_S(\mathcal{C}, (IS)) + i_S(\mathcal{C}, (\mathcal{J}S)) - \mu_S$. Again, in this case, one can check that $\delta_1(S, \mathcal{C}) = 1$. If $S \notin \ell_\infty$, we have

$$\text{class}(\Sigma_S(\mathcal{C})) = 13 - 2 \times \#\{i \in \{1, 2, 3, 4, 5, 6\} : S \in \ell_i \setminus \{a_i\}\} - g' \quad (7)$$

and if $S \in \ell_\infty \setminus \{I, \mathcal{J}\}$, we have

$$\text{class}(\Sigma_S(\mathcal{C})) = 11 - 3 \times \mathbf{1}_{S=A_2}. \quad (8)$$

We observe that the points of $\mathbb{P}^2 \setminus \{I, \mathcal{J}\}$ belonging to two distinct ℓ_k are outside \mathcal{C} . The set of these points is

$$\mathcal{E} := \bigcup_{k=1}^3 \left\{ \left[-\frac{3}{25} \sqrt[3]{20} \alpha^k : 0 : 1 \right], \left[\frac{3}{50} \sqrt[3]{20} \alpha^k : \frac{3}{50} \sqrt{3} \sqrt[3]{20} \alpha^k : 1 \right], \left[\frac{3}{50} \sqrt[3]{20} \alpha^k : -\frac{3}{50} \sqrt{3} \sqrt[3]{20} \alpha^k : 1 \right] \right\}$$

with $\alpha = e^{\frac{2i\pi}{3}}$. Finally, the class of the caustic in the different cases is summarized in the following table.

Condition on $S \in \mathbb{P}^2 \setminus \{I, \mathcal{J}\}$	$\text{class}(\Sigma_S(\mathcal{C})) =$
$S = A_2$	8
$S \in \mathcal{E}$	9
$S \in \mathcal{C} \cap \bigcup_{k=1}^6 (\ell_k \setminus \{a_k\})$	10
$S \in (\ell_\infty \setminus \{A_2\}) \cup \left(\bigcup_{k=1}^6 \ell_k \setminus (\mathcal{E} \cup \mathcal{C}) \right) \cup \{A_1\} \cup \{a_1, \dots, a_6\}$	11
$S \in \mathcal{C} \setminus \left(\ell_\infty \cup \{A_1\} \cup \bigcup_{k=1}^6 \ell_k \right)$	12
otherwise	13

5. ON THE FORMULAS BY BROCARD AND LEMOYNE AND BY SALMON AND CAYLEY

5.1. Formulas given by Brocard and Lemoyne. Recall that, when $S \notin \ell_\infty$, $\Sigma_S(\mathcal{C})$ is the evolute of an homothetic of the pedal of \mathcal{C} from S .

The work of Salmon and Cayley is under ordinary Plücker conditions (no hyper-flex, no singularities other than ordinary cups and ordinary nodes). In [18, p.137], Salmon and Cayley gave the following formula for the class of the evolute :

$$n' = m + n - f - g.$$

Replace now m , n , f and g by M , N , F and G (respectively) given in [18, p. 154] for the pedal. Doing so, one exactly get (with the same notations) the formula of the class of caustics by reflection given by Brocard and Lemoyne in [1, p. 114].

As explained in introduction, this composition of formulas of Salmon and Cayley is incorrect because of the non-conservation of the Plücker conditions by the pedal transformation. Nevertheless, for completeness sake, let us present the Brocard and Lemoyne formula and compare it with our formula. Brocard and Lemoyne gave the following formula for the class of the caustic by reflection $\Sigma_S(\mathcal{C})$ when $S \notin \ell_\infty$:

$$\text{class}(\Sigma_S(\mathcal{C})) = d + 2(d^\vee - \hat{f}') - \hat{g} - \hat{f} - \hat{g}' + \hat{q}', \quad (9)$$

for an algebraic curve \mathcal{C} of degree d , of class d^\vee , \hat{g} times tangent to ℓ_∞ , passing \hat{f} times through a cyclic point, \hat{f}' times tangent to an isotropic line of S , passing \hat{g}' times through S , \hat{q}' being the coincidence number of contact points when an isotropic line is multiply tangent. In [18], \hat{q}' is defined as the coincidence number of tangents at points ι_1, ι_2 of \mathbb{P}^{2^\vee} (corresponding to (IS) and

(JS) if these points are multiple points of the image of \mathcal{C} by the polar reciprocal transformation with center S ; i.e. \hat{q}' represents the number of ordinary flexes of \mathcal{C} .

When $S \notin \ell_\infty$, let us compare terms appearing in our formula (3) with terms of (9) :

- \hat{g} seems to be equal to g ;
- it seems that $\hat{f} = \mu_I + \mu_J$ and so

$$f = \hat{f} + \Omega_I(\mathcal{C}, (IS)) + \Omega_J(\mathcal{C}, (JS));$$

- it seems that $\hat{f}' = \sum_{m_1 \in \mathcal{C} \cap (IS)} \Omega_{m_1}(\mathcal{C}, (IS)) + \sum_{m_1 \in \mathcal{C} \cap (JS)} \Omega_{m_1}(\mathcal{C}, (JS))$ and so

$$f' := \hat{f}' - \Omega_I(\mathcal{C}, (IS)) - \Omega_J(\mathcal{C}, (JS)) - \Omega_S(\mathcal{C}, (IS)) - \Omega_S(\mathcal{C}, (JS));$$

- it seems that $\hat{g}' = \mu_S$, therefore

$$g' := \hat{g}' + \Omega_S(\mathcal{C}, (IS)) + \Omega_S(\mathcal{C}, (JS));$$

- our definition of q' appears as an extension of \hat{q}' (except that we exclude the points $m_1 \in \{I, J, S\}$).

Observe that these terms coincide with the definition of Brocard and Lemoyne if (IS) and (JS) are not tangent to \mathcal{C} at S, I, J . In particular, if we call BL the right hand side of (9), the first item Theorem 10 states that, when S is not at infinity we have

$$mclass(\Sigma_S(\mathcal{C})) = BL + \Omega_I(\mathcal{C}, (IS)) + \Omega_J(\mathcal{C}, (JS)) + \Omega_S(\mathcal{C}, (IS)) + \Omega_S(\mathcal{C}, (JS)).$$

5.2. A counterexample to the formula of Brocard and Lemoyne. We consider an example in which $\Omega_I(\mathcal{C}, (IS)) = \Omega_J(\mathcal{C}, (JS)) = 1$, which means that (IS) is tangent to \mathcal{C} at I and (JS) is tangent to \mathcal{C} at J . Let us consider the non-singular quartic curve $\mathcal{C} = V(2yz^3 + 2z^2y^2 + 2zy^3 + 2y^4 - 2z^3x + 2zyx^2 + 5y^2x^2 + 3x^4)$ and $S[0 : 0 : 1]$. This curve \mathcal{C} has degree $d = 4$ and class $d^\vee = 4 \times 3 = 12$, is not tangent to ℓ_∞ , is tangent to (SI) at I and nowhere else, is tangent to (SJ) at J and nowhere else; these tangent points are ordinary. S is a non singular point of \mathcal{C} . Therefore, with our definitions, we have $g = 0$, $f = 2 + 2 = 4$, $f' = 0$, $g' = 1 + 1 - 1 = 1$, $q' = 0$, which gives $class(\Sigma_S(\mathcal{C})) = 4 + 2(12 - 0) - 0 - 4 - 1 - 0 = 23$, since in this case $\delta_1(S, \mathcal{C}) = 1$. In comparison, the Brocard and Lemoyne formula would give $\hat{g} = 0$, $\hat{f} = 1 + 1 = 2$, $\hat{f}' = 1 + 1 = 2$, $\hat{g}' = 1$, $\hat{q}' = 0$ and so their formula gives $class(\Sigma_S(\mathcal{C})) = 4 + 2(12 - 2) - 0 - 2 - 1 - 0 = 21$ but this is false!

6. PROOF OF THEOREM 10

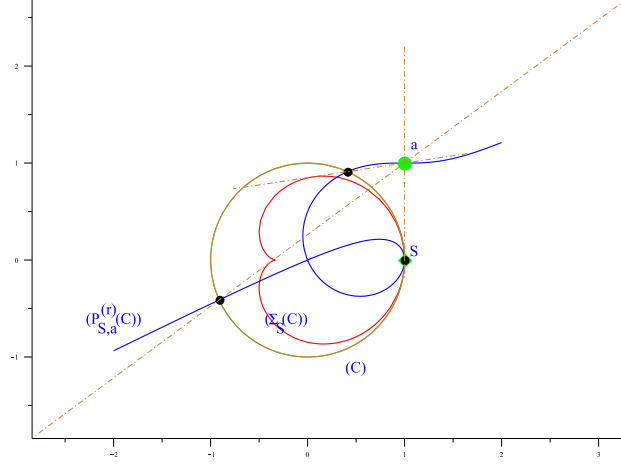
To compute the degree of $\overline{T_{\mathcal{C}, S}(\mathcal{C})}$, we will use the Fundamental Lemma given in [15]. Let us first recall the definition of φ -polar introduced in [15] and extending the notion of polar.

Definition 13. Let $p \geq 1$, $q \geq 1$ and let \mathbf{W} be a complex vector space of dimension $p + 1$. Given $\varphi : \mathbb{P}^p := \mathbb{P}(\mathbf{W}) \rightarrow \mathbb{P}^q$ a rational map defined by $\varphi = [\varphi_0 : \dots : \varphi_q]$ (with $\varphi_1, \dots, \varphi_q \in \text{Sym}^d(\mathbf{W}^\vee)$) and $a = [a_0 : \dots : a_q] \in \mathbb{P}^q$, we define the φ -polar at a , denoted by $\mathcal{P}_{\varphi, a}$, the hypersurface of degree d given by $\mathcal{P}_{\varphi, a} := V\left(\sum_{j=0}^q a_j \varphi_j\right) \subseteq \mathbb{P}^p$.

With this definition, the “classical” polar of a curve $\mathcal{C} = V(F)$ of \mathbb{P}^2 (for some homogeneous polynomial $F \in \mathbb{C}[x, y, z]$) at a is the $\delta_{\mathcal{C}}$ -polar curve at a , where $\delta_{\mathcal{C}} : [x : y : z] \mapsto [F_x : F_y : F_z]$.

Definition 14. We call **reflected polar (or r -polar) of the plane curve \mathcal{C} with respect to S at a** the $R_{\mathcal{C}, S}$ -polar at a , i.e. the curve $\mathcal{P}_{S, a}^{(r)}(\mathcal{C}) := \mathcal{P}_{R_{\mathcal{C}, S}, a}$.

From a geometric point of view, $\mathcal{P}_{S,a}^{(r)}(\mathcal{C})$ is an algebraic curve such that, for every $m \in \mathcal{C} \cap \mathcal{P}_{S,a}^{(r)}(\mathcal{C})$, \mathcal{R}_m contains a (if \mathcal{R}_m is well defined), this means that line (am) is tangent to $\Sigma_S(\mathcal{C})$ at the point $m' = \Phi_{F,S}(m) \in \Sigma_S(\mathcal{C})$ associated to m (see picture).



Let us now recall the statement of the fundamental lemma proved in [15].

Lemma 15 (Fundamental lemma [15]). *Let \mathbf{W} be a complex vector space of dimension $p + 1$, let \mathcal{C} be an irreducible algebraic curve of $\mathbb{P}^p := \mathbb{P}(\mathbf{W})$ and $\varphi : \mathbb{P}^p \rightarrow \mathbb{P}^q$ be a rational map given by $\varphi = [\varphi_0 : \cdots : \varphi_q]$ with $\varphi_0, \dots, \varphi_q \in \text{Sym}^\delta(\mathbf{W}^\vee)$. Assume that $\mathcal{C} \not\subseteq \text{Base}(\varphi)$ and that $\varphi|_{\mathcal{C}}$ has degree $\delta_1 \in \mathbb{N} \cup \{\infty\}$. Then, for generic $a = [a_0 : \cdots : a_q] \in \mathbb{P}^q$, the following formula holds true*

$$\delta_1 \cdot \deg(\overline{\varphi(\mathcal{C})}) = \delta \cdot \deg(\mathcal{C}) - \sum_{p \in \text{Base}(\varphi|_{\mathcal{C}})} i_p(\mathcal{C}, \mathcal{P}_{\varphi,a}),$$

with convention $0 \cdot \infty = 0$ and $\deg(\overline{\varphi(\mathcal{C})}) = 0$ if $\#\overline{\varphi(\mathcal{C})} < \infty$.

Due to this lemma and to Proposition 9, we have

$$\text{mclass}(\Sigma_S(\mathcal{C})) = d(2d - 1) - \sum_{m_1 \in \text{Base}(T_{\mathcal{C},S})} i_{m_1}(\mathcal{C}, \mathcal{P}_{S,a}^{(r)}(\mathcal{C})). \quad (10)$$

Now, we enter in the most technical stuff which is the computation of the intersection numbers $i_{m_1}(\mathcal{C}, \mathcal{P}_{S,a}^{(r)}(\mathcal{C}))$ of \mathcal{C} with its reflected polar at the base points of $\mathcal{R}_{\mathcal{C},S}$. To compute these intersection numbers, it will be useful to observe the form of the image of $\mathcal{R}_{\mathcal{C},S}$ by linear changes of variable. It is worth noting that $\mathbf{R}_{F,S}$ can be rewritten

$$\mathbf{R}_{F,S} = id \wedge [\Delta_{\mathbf{I}} F \Delta_{\mathbf{J}} F \cdot \mathbf{S} - \Delta_{\mathbf{S}} F \Delta_{\mathbf{I}} F \cdot \mathbf{J} - \Delta_{\mathbf{S}} F \Delta_{\mathbf{J}} F \cdot \mathbf{I}].$$

Proposition 16. *Let $M \in GL(\mathbf{V})$. We have*

$$\mathbf{R}_{F,\mathbf{S}} \circ M = Com(M) \cdot \mathbf{R}_{F \circ M, M^{-1}(\mathbf{S})}^{(M^{-1}(\mathbf{I}), M^{-1}(\mathbf{J}))},$$

with $Com(M) := det(M) \cdot {}^t M^{-1}$ and

$$\mathbf{R}_{G,\mathbf{S}'}^{(\mathbf{A},\mathbf{B})} := id \wedge [\Delta_{\mathbf{A}} G \Delta_{\mathbf{B}} G \cdot \mathbf{S}' - \Delta_{\mathbf{S}'} G \Delta_{\mathbf{A}} G \cdot \mathbf{B} - \Delta_{\mathbf{S}'} G \Delta_{\mathbf{B}} G \cdot \mathbf{A}].$$

Proof. We use $M(\mathbf{u}) \wedge M(\mathbf{v}) = (Com(M))(\mathbf{u} \wedge \mathbf{v})$ and $\Delta_{M(\mathbf{u})}(F)(M(\mathbf{P})) = \Delta_{\mathbf{u}}(F \circ M)(\mathbf{P})$. \square

We write $\Pi : \mathbf{V} \setminus \{\mathbf{0}\} \rightarrow \mathbb{P}^2$ for the canonical projection, $P_0[0 : 0 : 1] \in \mathbb{P}^2$ and $\mathbf{P}_0(0, 0, 1) \in \mathbf{V}$. Let m_1 be a base point of \mathcal{C} and $M \in GL(\mathbf{V})$ be such that $\Pi(M(\mathbf{P}_0)) = m_1$ and such that the tangent cone of $V(F \circ M)$ at P_0 does not contain $V(x)$. Let μ_{m_1} be the multiplicity of m_1 in \mathcal{C} (m_1 is a singular point of \mathcal{C} if and only if $\mu_{m_1} > 1$). Then, for every $a \in \mathbb{P}^2$, writing $a' := M^{-1}(a)$, we have

$$\begin{aligned} i_{m_1}(\mathcal{C}, \mathcal{P}_{S,a}^{(r)}(\mathcal{C})) &= i_{m_1}(\mathcal{C}, V(\langle \mathbf{a}, \mathbf{R}_{F,\mathbf{S}}(\cdot) \rangle)) \\ &= i_{P_0}(V(F \circ M), V(\langle \mathbf{a}, \mathbf{R}_{F,\mathbf{S}} \circ M(\cdot) \rangle)) \\ &= i_{P_0}(V(F \circ M), V(\langle \mathbf{a}', \mathbf{R}_{F \circ M, M^{-1}(\mathbf{S})}^{(M^{-1}(\mathbf{I}), M^{-1}(\mathbf{J}))}(\cdot) \rangle)) \\ &= \sum_{\mathcal{B} \in Branch_{P_0}(V(F \circ M))} i_{P_0}(\mathcal{B}, V(\langle \mathbf{a}', \mathbf{R}_{F \circ M, M^{-1}(\mathbf{S})}^{(M^{-1}(\mathbf{I}), M^{-1}(\mathbf{J}))}(\cdot) \rangle)), \end{aligned}$$

where $Branch_{P_0}(V(F \circ M))$ is the set of branches of $V(F \circ M)$ at P_0 . The last equality comes from Proposition 18 proved in appendix (see formula (14)). Let b be the number of such branches. Of course, $b = 1$ for non-singular points. Writing $e_{\mathcal{B}}$ for the multiplicity of the branch \mathcal{B} , we have $\mu_{m_1} = \sum_{\mathcal{B} \in Branch_{P_0}(V(F \circ M))} e_{\mathcal{B}}$. Let us write $\mathbb{C}\langle x^{\frac{1}{N}} \rangle$ and $\mathbb{C}\langle x^{\frac{1}{N}}, y \rangle$ for the rings of convergent power series of $x^{\frac{1}{N}}, y$. Let $\mathbb{C}\langle x^* \rangle := \bigcup_{N \geq 1} \mathbb{C}\langle x^{\frac{1}{N}} \rangle$ and $\mathbb{C}\langle x^*, y \rangle := \bigcup_{N \geq 1} \mathbb{C}\langle x^{\frac{1}{N}}, y \rangle$. For every $h = \sum_{q \in \mathbb{Q}_+} a_q x^q \in \mathbb{C}\langle x^* \rangle$, we define the valuation of h as follows:

$$val(h) := val_x(h(x)) := \min\{q \in \mathbb{Q}_+, a_q \neq 0\}.$$

Let \mathcal{B} be a branch of $V(F \circ M)$ at P_0 . We precise that $\mathcal{B}_0 = M(\mathcal{B}) \subset \mathbb{P}^2$ is a branch of \mathcal{C} at m_1 . Let $\mathbf{A}(x_A, y_A, z_A) := M^{-1}(\mathbf{I})$, $\mathbf{B}(x_B, y_B, z_B) := M^{-1}(\mathbf{J})$ and $\mathbf{S}' := M^{-1}(\mathbf{S})$. Let $\mathcal{T}_{\mathcal{B}}$ be the tangent line to \mathcal{B} at P_0 . The branch \mathcal{B} can be splitted in $e_{\mathcal{B}}$ pro-branches with equations $y = g_{i,\mathcal{B}}(x)$ in the chart $z = 1$ (for $i \in \{1, \dots, e_{\mathcal{B}}\}$) with $g_i \in \mathbb{C}\langle x^* \rangle$ having (rational) valuation larger than or equal to 1 (so $g'_i(0) = 0$). For $j \in \{1, \dots, e_{\mathcal{B}'}\}$, consider also the equations $y = g_{j,\mathcal{B}'}(x)$ (in the chart $z = 1$) of the pro-branches $\mathcal{V}_{j,\mathcal{B}'}$ for each branch $\mathcal{B}' \in Branch_{P_0}(V(F \circ M))$. This notion of pro-branches comes from the combination of the Weierstrass and of the Puiseux theorems. It has been used namely by Halphen in [13] and by Wall in [21]. One can also see [15]. There exists a unit U of $\mathbb{C}\langle x, y \rangle$ such that the following equality holds true in $\mathbb{C}\langle x^*, y \rangle$

$$F(M(x, y, 1)) = U(x, y) \prod_{\mathcal{B}' \in Branch_{P_0}(V(F \circ M))} \prod_{j=1}^{e_{\mathcal{B}'}} (y - g_{j,\mathcal{B}'}(x)).$$

For a generic a (with $a' := M^{-1}(a)$), using (15)), we obtain

$$\begin{aligned} i_{P_0}(\mathcal{B}, V(\langle \mathbf{a}', \mathbf{R}_{F \circ M, \mathbf{S}'}^{(\mathbf{A}, \mathbf{B})}(\cdot) \rangle)) &= \sum_i val_x \left(\langle \mathbf{a}', \mathbf{R}_{F \circ M, \mathbf{S}'}^{(\mathbf{A}, \mathbf{B})}(x, g_{i,\mathcal{B}}(x), 1) \rangle \right) \\ &= \sum_i \min_{j=1,2,3} val_x \left(\left[\mathbf{R}_{F \circ M, \mathbf{S}'}^{(\mathbf{A}, \mathbf{B})}(x, g_{i,\mathcal{B}}(x), 1) \right]_j \right). \end{aligned}$$

Hence Formula (10) becomes

$$\text{mclass}(\Sigma_S(\mathcal{C})) := d(2d-1) - \sum_{m_1 \in \mathcal{C}} \sum_{\mathcal{B}} \sum_{i=1}^{e_{\mathcal{B}}} \min_{j=1,2,3} \text{val}_x \left(\left[\mathbf{R}_{F \circ M, \mathbf{S}'}^{(\mathbf{A}, \mathbf{B})}(\mathbf{x}, \mathbf{g}_{i, \mathcal{B}}(\mathbf{x}), \mathbf{1}) \right]_j \right), \quad (11)$$

where, for every $m_1 \in \mathcal{C}$, M depends on m_1 and is as above, where the sum is over $\mathcal{B} \in \text{Branch}_{P_0}(V(F \circ M))$. Due to Lemma 33 of [15], for every $\mathbf{P}(x_P, y_P, z_P) \in \mathbf{V} \setminus \{\mathbf{0}\}$, we have

$$(\Delta_M(\mathbf{P})F) \circ M(x, g_{i, \mathcal{B}}(x), 1) = \Delta_{\mathbf{P}}(F \circ M)(x, g_{i, \mathcal{B}}(x), 1) = D_{i, \mathcal{B}}(x) W_{\mathbf{P}, i, \mathcal{B}}(x),$$

with

$$W_{\mathbf{P}, i, \mathcal{B}}(x) := y_P - g'_{i, \mathcal{B}}(x) x_P + z_P (x g'_{i, \mathcal{B}}(x) - g_{i, \mathcal{B}}(x))$$

and with $D_{i, \mathcal{B}}(x) := U(x, g_{i, \mathcal{B}}(x)) \prod_{\mathcal{B}' \in \text{Branch}_{P_0}(V(F \circ M))} \prod_{j=1, \dots, e_{\mathcal{B}'}: (\mathcal{B}', j) \neq (\mathcal{B}, i)} (g_{i, \mathcal{B}}(x) - g_{j, \mathcal{B}'}(x))$. Hence we have

$$\mathbf{R}_{F \circ M, \mathbf{S}'}^{(\mathbf{A}, \mathbf{B})}(x, g_{i, \mathcal{B}}(x), 1) := (D_{i, \mathcal{B}})^2 \cdot \hat{R}_{i, \mathcal{B}}(x) \quad (12)$$

with

$$\hat{R}_{i, \mathcal{B}}(x) := \begin{pmatrix} x \\ g_{i, \mathcal{B}}(x) \\ 1 \end{pmatrix} \wedge [W_{\mathbf{A}, i, \mathcal{B}}(x) W_{\mathbf{B}, i, \mathcal{B}}(x) \cdot \mathbf{S}' - W_{\mathbf{S}', i, \mathcal{B}}(x) W_{\mathbf{A}, i, \mathcal{B}}(x) \cdot \mathbf{B} - W_{\mathbf{S}', i, \mathcal{B}}(x) W_{\mathbf{B}, i, \mathcal{B}}(x) \cdot \mathbf{A}].$$

First, with the notations of [15] (since $U(0, 0) \neq 0$), we have

$$\sum_{\mathcal{B} \in \text{Branch}_{P_0}(V(F \circ M))} \sum_{i=1}^{e_{\mathcal{B}}} \text{val}(D_{i, \mathcal{B}}) = V_{m_1},$$

(which is null if m_1 is a nonsingular point of \mathcal{C}). Second, writing $h_{m_1, i, \mathcal{B}} := \min(\text{val}([\hat{R}_{i, \mathcal{B}}]_j))$, $j = 1, 2, 3$, we observe that, due to Proposition 29 and to Remark 34 of [15], the quantity $\sum_{i=1}^{e_{\mathcal{B}}} h_{m_1, i, \mathcal{B}}$ only depends on m_1 and on the branch $\mathcal{B}_0 = M(\mathcal{B})$ of \mathcal{C} at m_1 (it does not depend on the choice of $M \in GL(\mathbf{V})$ such that $\Pi(M(P_0)) = m_1$ and such that $V(x)$ is not tangent to $M^{-1}(\mathcal{B}_0)$). Hence we write

$$h_{m_1, \mathcal{B}_0} := \sum_{i=1}^{e_{\mathcal{B}}} h_{m_1, i, \mathcal{B}}.$$

With these notations, due to (12), formula (11) becomes

$$\text{mclass}(\Sigma_S(\mathcal{C})) = 2d(d-1) + d - 2 \sum_{m_1 \in \text{Sing}(\mathcal{C})} V_{m_1} - \sum_{m_1 \in \mathcal{C}} \sum_{\mathcal{B}_0 \in \text{Branch}_{m_1}(\mathcal{C})} h_{m_1, \mathcal{B}_0}.$$

Moreover, as noticed in [15], we have $d(d-1) - \sum_{m_1 \in \text{Sing}(\mathcal{C})} V_{m_1} = d^\vee$, where d^\vee is the class of \mathcal{C} . Therefore, we get

$$\text{mclass}(\Sigma_S(\mathcal{C})) = 2d^\vee + d - \sum_{m_1 \in \mathcal{C}} \sum_{\mathcal{B}_0 \in \text{Branch}_{m_1}(\mathcal{C})} h_{m_1, \mathcal{B}_0}. \quad (13)$$

Theorem 10 will come directly from the computation of $h_{m_1, i, \mathcal{B}}$ given in following result.

Lemma 17. *Let $m_1 \in \mathcal{C}$ and $\mathcal{B}_0 \in \text{Branch}_{m_1}(\mathcal{C})$. Writing $\mathcal{T}_{m_1} \mathcal{B}_0$ for the tangent line to \mathcal{B}_0 at m_1 , $i_{m_1}(\mathcal{B}_0, \mathcal{T}_{m_1} \mathcal{B}_0)$ for the intersection number of \mathcal{B}_0 with $\mathcal{T}_{m_1} \mathcal{B}_0$ at m_1 and $e_{\mathcal{B}_0}$ for the multiplicity of \mathcal{B}_0 , we have*

- (1) $h_{m_1, \mathcal{B}_0} = 0$ if $I, J, S \notin \mathcal{T}_{m_1} \mathcal{B}_0$.
- (2) $h_{m_1, \mathcal{B}_0} = 0$ if $\#(\mathcal{T}_{m_1} \mathcal{B}_0 \cap \{I, J, S\}) = 1$ and $m_1 \notin \{I, J, S\}$.
- (3) $h_{m_1, \mathcal{B}_0} = e_{\mathcal{B}_0}$ if $\#(\mathcal{T}_{m_1} \mathcal{B}_0 \cap \{I, J, S\}) = 1$ and $m_1 \in \{I, J, S\}$.

- (4) $h_{m_1, \mathcal{B}_0} = i_{m_1}(\mathcal{B}_0, \mathcal{T}_{m_1}\mathcal{B}_0) + \min(i_{m_1}(\mathcal{B}_0, \mathcal{T}_{m_1}\mathcal{B}_0) - 2e_{\mathcal{B}_0}, 0)$ if $\mathcal{T}_{m_1}\mathcal{B}_0 = (IS)$, $J \notin \mathcal{T}_{m_1}\mathcal{B}_0$ and $m_1 \notin \{I, S\}$.
 $h_{m_1, \mathcal{B}_0} = i_{m_1}(\mathcal{B}_0, \mathcal{T}_{m_1}\mathcal{B}_0) + \min(i_{m_1}(\mathcal{B}_0, \mathcal{T}_{m_1}\mathcal{B}_0) - 2e_{\mathcal{B}_0}, 0)$ if $\mathcal{T}_{m_1}\mathcal{B}_0 = (JS)$, $I \notin \mathcal{T}_{m_1}\mathcal{B}_0$ and $m_1 \notin \{J, S\}$.
- (5) $h_{m_1, \mathcal{B}_0} = i_{m_1}(\mathcal{B}_0, \mathcal{T}_{m_1}\mathcal{B}_0)$ if $\mathcal{T}_{m_1}\mathcal{B}_0 = (IS)$, $J \notin \mathcal{T}_{m_1}\mathcal{B}_0$ and $m_1 \in \{I, S\}$.
 $h_{m_1, \mathcal{B}_0} = i_{m_1}(\mathcal{B}_0, \mathcal{T}_{m_1}\mathcal{B}_0)$ if $\mathcal{T}_{m_1}\mathcal{B}_0 = (JS)$, $I \notin \mathcal{T}_{m_1}\mathcal{B}_0$ and $m_1 \in \{J, S\}$.
- (6) $h_{m_1, \mathcal{B}_0} = i_{m_1}(\mathcal{B}_0, \mathcal{T}_{m_1}\mathcal{B}_0) - e_{\mathcal{B}_0}$ if $\mathcal{T}_{m_1}\mathcal{B}_0 = (IJ)$, that $S \notin \mathcal{T}_{m_1}\mathcal{B}_0$ and $m_1 \notin \{I, J\}$.
- (7) $h_{m_1, \mathcal{B}_0} = i_{m_1}(\mathcal{B}_0, \mathcal{T}_{m_1}\mathcal{B}_0)$ if $\mathcal{T}_{m_1}\mathcal{B}_0 = (IJ)$, that $S \notin \mathcal{T}_{m_1}\mathcal{B}_0$ and $m_1 \in \{I, J\}$.
- (8) $h_{m_1, \mathcal{B}_0} = 2i_{m_1}(\mathcal{B}_0, \mathcal{T}_{m_1}\mathcal{B}_0) - 2e_{\mathcal{B}_0}$ if $I, J, S \in \mathcal{T}_{m_1}\mathcal{B}_0$ and $m_1 \notin \{I, J, S\}$.
- (9) $h_{m_1, \mathcal{B}_0} = 2i_{m_1}(\mathcal{B}_0, \mathcal{T}_{m_1}\mathcal{B}_0) - e_{\mathcal{B}_0}$ if $I, J, S \in \mathcal{T}_{m_1}\mathcal{B}_0$ and $m_1 \in \{I, J\}$.
- (10) $h_{m_1, \mathcal{B}_0} = 2i_{m_1}(\mathcal{B}_0, \mathcal{T}_{m_1}\mathcal{B}_0) - e_{\mathcal{B}_0}$ if $I, J, S \in \mathcal{T}_{m_1}\mathcal{B}_0$, $m_1 = S$ and $i_{m_1}(\mathcal{B}_0, \mathcal{T}_{m_1}\mathcal{B}_0) \neq 2e_{\mathcal{B}_0}$.
- (11) $h_{m_1, \mathcal{B}_0} = e_{\mathcal{B}_0}(1 + \min(\beta_1, 3))$ if $I, J, S \in \mathcal{T}_{m_1}\mathcal{B}_0$, $m_1 = S$ and $i_{m_1}(\mathcal{B}_0, \mathcal{T}_{m_1}\mathcal{B}_0) = 2e_{\mathcal{B}_0}$, $e_{\mathcal{B}_0}\beta_1 = i_{m_1}(\mathcal{B}_0, \text{Osc}_{m_1}(\mathcal{B}_0))$, where $\text{Osc}_{m_1}(\mathcal{B}_0)$ is any osculating smooth algebraic curve to \mathcal{B}_0 at m_1 (the last formula of h_{m_1, \mathcal{B}_0} holds true if we replace $e_{\mathcal{B}_0}\beta_1$ by the first characteristic exponent of \mathcal{B}_0 non multiple of $e_{\mathcal{B}_0}$, see [22]).

Proof. We take M such that $\mathcal{T}_{\mathcal{B}} = V(y)$ (with $\mathcal{B} = M^{-1}(\mathcal{B}_0)$). To simplify notations, we omit indices \mathcal{B} in $W_{\mathbf{P}, i, \mathcal{B}}$ and consider $i \in \{1, \dots, e_{\mathcal{B}}\}$.

- Suppose that $I, J, S \notin \mathcal{T}_{m_1}\mathcal{B}_0$. Then $W_{\mathbf{B}, i}(0) = y_B \neq 0$, $W_{\mathbf{A}, i}(0) = y_A \neq 0$ and $W_{\mathbf{S}', i}(0) = y_{S'} \neq 0$ so

$$\begin{aligned} \hat{R}_i(0) &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \wedge [y_A y_B \cdot \mathbf{S}' - y_A y_{S'} \cdot \mathbf{B} - y_B y_{S'} \cdot \mathbf{A}] \\ &= \begin{pmatrix} y_A y_B y_{S'} \\ y_A y_B x_{S'} - y_A y_{S'} x_B - y_B y_{S'} x_A \\ 0 \end{pmatrix}. \end{aligned}$$

Hence $h_{m_1, i, \mathcal{B}} = 0$ and the sum over $i = 1, \dots, e_{\mathcal{B}}$ of these quantities is equal to 0.

- Suppose $I \in \mathcal{T}_{m_1}\mathcal{B}_0$, $J, S \notin \mathcal{T}_{m_1}\mathcal{B}_0$ and $m_1 \neq I$. Take M such that $\mathbf{S}'(0, 1, 0)$, $\mathbf{A}(1, 0, 0)$, $y_B \neq 0$. We have $W_{\mathbf{B}, i}(0) = y_B$, $W_{\mathbf{A}, i}(0) = 0$ and $W_{\mathbf{S}', i}(0) = 1$ and so $\hat{R}_i(0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \wedge \begin{pmatrix} -y_B \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -y_B \\ 0 \end{pmatrix}$. Hence $h_{m_1, i, \mathcal{B}} = 0$ and the sum over $i = 1, \dots, e_{\mathcal{B}}$ of these quantities is equal to 0.
- Suppose $I \in \mathcal{T}_{m_1}\mathcal{B}_0$, $J, S \notin \mathcal{T}_{m_1}\mathcal{B}_0$ and $m_1 = I$. Take M such that $\mathbf{S}'(0, 1, 0)$, $\mathbf{A}(0, 0, 1)$, $y_B \neq 0$. We have $W_{\mathbf{B}, i}(x) = y_B - g'_i(x)x_B + z_B(xg'_i(x) - g_i(x))$, $W_{\mathbf{A}, i}(x) = xg'_i(x) - g_i(x)$ and $W_{\mathbf{S}', i}(x) = 1$ and so

$$\begin{aligned} \hat{R}_i(x) &= \begin{pmatrix} x \\ g_i(x) \\ 1 \end{pmatrix} \wedge \begin{pmatrix} -(xg'_i(x) - g_i)x_B \\ (xg'_i(x) - g_i)(-g'_i(x)x_B + z_B(xg'_i(x) - g_i(x))) \\ -y_B + g'_i(x)x_B - 2z_B(xg'_i(x) - g_i(x)) \end{pmatrix} \\ &= \begin{pmatrix} -y_B g_i(x) + x(g'_i(x))^2 x_B - z_B((xg'_i(x))^2 - (g_i(x))^2) \\ -x_B(2xg'_i(x) - g_i(x)) + xy_B + 2xz_B(xg'_i(x) - g_i(x)) \\ -x_B(xg'_i(x) - g_i(x))^2 + z_B(xg'_i(x) - g_i(x)) \end{pmatrix}, \end{aligned}$$

the valuation of the coordinates of which are larger than or equal to 1 and the valuation of the second coordinate is 1. Hence $h_{m_1, i, \mathcal{B}} = 1$ and the sum over $i = 1, \dots, e_{\mathcal{B}} = e_{\mathcal{B}_0}$ of these quantities is equal to $e_{\mathcal{B}_0}$.

- Suppose $S \in \mathcal{T}_{m_1}\mathcal{B}_0$, $I, J \notin \mathcal{T}_{m_1}\mathcal{B}_0$ and $m_1 \neq S$. Take M such that $\mathbf{A}(0, 1, 0)$, $\mathbf{S}'(1, 0, 0)$, $y_B \neq 0$. We have $W_{\mathbf{B}, i}(0) = y_B \neq 0$, $W_{\mathbf{S}', i}(0) = 0$ and $W_{\mathbf{A}, i}(0) = 1$ and so $\hat{R}_i(0) =$

$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \wedge \begin{pmatrix} y_B \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ y_B \\ 0 \end{pmatrix}$. Hence $h_{m_1, i, \mathcal{B}} = 0$ and the sum over $i = 1, \dots, e_{\mathcal{B}}$ of these quantities is equal to 0.

- Suppose $m_1 = S$ and $I, J \notin \mathcal{T}_{m_1} \mathcal{B}_0$. Take M such that $\mathbf{S}'(0, 0, 1)$, $\mathbf{A}(0, 1, 0)$, $y_B \neq 0$. We have $W_{\mathbf{B}, i}(x) = y_B - g'_i(x)x_B + z_B(xg'_i(x) - g_i(x))$, $W_{\mathbf{S}', i}(x) = xg'_i(x) - g_i(x)$ and $W_{\mathbf{A}, i}(x) = 1$ and so

$$\begin{aligned} \hat{R}_i(x) &= \begin{pmatrix} x \\ g_i(x) \\ 1 \end{pmatrix} \wedge \begin{pmatrix} -(xg'_i(x) - g_i(x))x_B \\ -(xg'_i(x) - g_i(x))(2y_B - g'_i(x)x_B + z_B(xg'_i(x) - g_i(x))) \\ y_B - g'_i(x)x_B \end{pmatrix} \\ &= \begin{pmatrix} g_i(x)(y_B - g'_i(x)x_B) + (xg'_i(x) - g_i(x))(2y_B - g'_i(x)x_B + z_B(xg'_i(x) - g_i(x))) \\ -(xg'_i(x) - g_i(x))x_B - x(y_B - g'_i(x)x_B) \\ -(xg'_i(x) - g_i(x))(2xy_B - g'_i(x)xx_B + z_Bx(xg'_i(x) - g_i(x)) + g_i(x)(xg'_i(x) - g_i(x))x_B) \end{pmatrix}, \end{aligned}$$

the valuation of the coordinates of which are larger than or equal to 1 and the valuation of the second coordinate is 1. Hence $h_{m_1, i, \mathcal{B}} = 1$ and the sum over $i = 1, \dots, e_{\mathcal{B}}$ of these quantities is equal to $e_{\mathcal{B}_0}$.

- Suppose $\mathcal{T}_{m_1} \mathcal{B}_0 = (IS)$, $J \notin \mathcal{T}_{m_1} \mathcal{B}_0$ and $m_1 \notin \{I, S\}$. Take M such that $\mathbf{S}'(1, 0, 0)$, $\mathbf{B}(0, 1, 0)$, $y_A = 0$, $x_A \neq 0$, $z_A \neq 0$. We have $W_{\mathbf{S}', i}(x) = -g'_i(x)$, $W_{\mathbf{A}, i}(x) = -g'_i(x)x_A + z_A(xg'_i(x) - g_i(x))$ and $W_{\mathbf{B}, i}(x) = 1$ and so

$$\begin{aligned} \hat{R}_i(x) &= \begin{pmatrix} x \\ g_i(x) \\ 1 \end{pmatrix} \wedge \begin{pmatrix} z_A(xg'_i(x) - g_i(x)) \\ -(g'_i(x))^2x_A + g'_i(x)(xg'_i(x) - g_i(x))z_A \\ g'_i(x)z_A \end{pmatrix} \\ &= \begin{pmatrix} g_i(x)g'_i(x)z_A + (g'_i(x))^2x_A - g'_i(x)(xg'_i(x) - g_i(x))z_A \\ -g_i(x)z_A \\ -x(g'_i(x))^2x_A + (xg'_i(x) - g_i(x))^2z_A \end{pmatrix}, \end{aligned}$$

the valuation of the coordinates of which are respectively $2\text{val}(g_i) - 2$, $\text{val}(g_i)$ and $2\text{val}(g_i) - 1$. Hence $h_{m_1, i, \mathcal{B}} = \text{val}(g_i) + \min(\text{val}(g_i) - 2, 0)$ and the sum over $i = 1, \dots, e_{\mathcal{B}}$ of these quantities is equal to $i_{m_1}(\mathcal{B}_0, \mathcal{T}_{m_1} \mathcal{B}_0) + \min(i_{m_1}(\mathcal{B}_0, \mathcal{T}_{m_1} \mathcal{B}_0) - 2e_{\mathcal{B}_0}, 0)$.

- Suppose $\mathcal{T}_{m_1} \mathcal{B}_0 = (IS)$, $J \notin \mathcal{T}_{m_1} \mathcal{B}_0$ and $m_1 = I$. Take M such that $\mathbf{S}'(1, 0, 0)$, $\mathbf{B}(0, 1, 0)$, $\mathbf{A}(0, 0, 1)$. We have $W_{\mathbf{S}', i}(x) = -g'_i(x)$, $W_{\mathbf{A}, i}(x) = xg'_i(x) - g_i(x)$ and $W_{\mathbf{B}, i}(x) = 1$ and so

$$\hat{R}_i(x) = \begin{pmatrix} x \\ g_i(x) \\ 1 \end{pmatrix} \wedge \begin{pmatrix} xg'_i(x) - g_i(x) \\ g'_i(x)(xg'_i(x) - g_i(x)) \\ g'_i(x) \end{pmatrix} = \begin{pmatrix} g'_i(x)(2g_i(x) - xg'_i(x)) \\ -g_i(x) \\ (xg'_i(x) - g_i(x))^2 \end{pmatrix},$$

the valuation of the coordinates of which are larger than or equal to $\text{val}(g_i)$, the second coordinate has valuation $\text{val}(g_i)$. Hence $h_{m_1, i, \mathcal{B}} = \text{val}(g_i)$ and the sum over $i = 1, \dots, e_{\mathcal{B}}$ of these quantities is equal to $i_{m_1}(\mathcal{B}_0, \mathcal{T}_{m_1} \mathcal{B}_0)$.

- Suppose $\mathcal{T}_{m_1} \mathcal{B}_0 = (IS)$, $J \notin \mathcal{T}_{m_1} \mathcal{B}_0$ and $m_1 = S$. Take M such that $\mathbf{A}(1, 0, 0)$, $\mathbf{B}(0, 1, 0)$, $\mathbf{S}'(0, 0, 1)$. We have $W_{\mathbf{S}', i}(x) = xg'_i(x) - g_i(x)$, $W_{\mathbf{A}, i}(x) = -g'_i(x)$ and $W_{\mathbf{B}, i}(x) = 1$ and so

$$\hat{R}_i(x) = \begin{pmatrix} x \\ g_i(x) \\ 1 \end{pmatrix} \wedge \begin{pmatrix} -(xg'_i(x) - g_i(x)) \\ g'_i(x)(xg'_i(x) - g_i(x)) \\ -g'_i(x) \end{pmatrix} = \begin{pmatrix} -x(g'_i(x))^2 \\ g_i(x) \\ (xg'_i(x))^2 - (g_i(x))^2 \end{pmatrix},$$

the valuation of the coordinates of which being larger than or equal to $\text{val}(g_i)$ and the valuation of the second coordinate is equal to $\text{val}(g_i)$. Hence $h_{m_1, i, \mathcal{B}} = \text{val}(g_i)$ and the sum over $i = 1, \dots, e_{\mathcal{B}}$ of these quantities is equal to $i_{m_1}(\mathcal{B}_0, \mathcal{T}_{m_1} \mathcal{B}_0)$.

- Suppose $\mathcal{T}_{m_1}\mathcal{B}_0 = (IJ)$, $S \notin \mathcal{T}_{m_1}\mathcal{B}_0$ and $m_1 \notin \{I, J\}$. Take M such that $\mathbf{S}'(0, 1, 0)$, $\mathbf{B}(1, 0, 0)$, $y_A = 0$, $x_A \neq 0$, $z_A \neq 0$. We have $W_{\mathbf{B},i}(x) = -g'_i(x)$, $W_{\mathbf{A},i}(x) = -g'_i(x)x_A + z_A(xg'_i(x) - g_i(x))$ and $W_{\mathbf{S}',i}(x) = 1$ and so

$$\begin{aligned}\hat{R}_i(x) &= \begin{pmatrix} x \\ g_i(x) \\ 1 \end{pmatrix} \wedge \begin{pmatrix} 2g'_i(x)x_A - z_A(xg'_i(x) - g_i(x)) \\ (g'_i(x))^2x_A - g'_i(x)(xg'_i(x) - g_i(x))z_A \\ g'_i(x)z_A \end{pmatrix} \\ &= \begin{pmatrix} (g'_i(x))^2(xz_A - x_A) \\ 2g'_i(x)x_A - z_A(2xg'_i(x) - g_i(x)) \\ -z_A(xg'_i(x) - g_i(x))^2 + x_Ag'_i(x)(xg'_i(x) - 2g_i(x)) \end{pmatrix},\end{aligned}$$

the valuation of the coordinates of which are respectively $2\text{val}(g_i) - 2$, $\text{val}(g_i) - 1$ and larger than $\text{val}(g_i)$. Hence $h_{m_1,i,\mathcal{B}} = \text{val}(g_i) - 1$ and the sum over $i = 1, \dots, e_{\mathcal{B}}$ of these quantities is equal to $i_{m_1}(\mathcal{B}_0, \mathcal{T}_{m_1}\mathcal{B}_0) - e_{\mathcal{B}_0}$.

- Suppose that $\mathcal{T}_{m_1}\mathcal{B}_0 = (IJ)$, that $S \notin \mathcal{T}_{m_1}\mathcal{B}_0$ and $m_1 = I$. Take M such that $\mathbf{S}'(0, 1, 0)$, $\mathbf{B}(1, 0, 0)$, $\mathbf{A}(0, 0, 1)$. We have $W_{\mathbf{B},i}(x) = -g'_i(x)$, $W_{\mathbf{A},i}(x) = xg'_i(x) - g_i(x)$ and $W_{\mathbf{S}',i}(x) = 1$ and so

$$\hat{R}_i(x) = \begin{pmatrix} x \\ g_i(x) \\ 1 \end{pmatrix} \wedge \begin{pmatrix} -(xg'_i(x) - g_i(x)) \\ -g'_i(x)(xg'_i(x) - g_i(x)) \\ g'_i(x) \end{pmatrix} = \begin{pmatrix} x(g'_i(x))^2 \\ -(2xg'_i(x) - g_i(x)) \\ -(xg'_i(x) - g_i(x))^2 \end{pmatrix},$$

the valuation of the coordinates of which being larger than or equal to $\text{val}(g_i)$ and the valuation of the second coordinate is equal to $\text{val}(g_i)$. Hence $h_{m_1,i,\mathcal{B}} = \text{val}(g_i)$ and the sum over $i = 1, \dots, e_{\mathcal{B}}$ of these quantities is equal to $i_{m_1}(\mathcal{B}_0, \mathcal{T}_{m_1}\mathcal{B}_0)$.

- Suppose that $I, J, S \in \mathcal{T}_{m_1}\mathcal{B}_0$ and $m_1 \notin \{I, J, S\}$. Take M such that $\mathbf{S}'(1, 0, 0)$, $y_A = y_B = 0$, $x_A \neq 0$, $z_A \neq 0$, $x_B \neq 0$, $z_B \neq 0$, $x_Az_B \neq x_Bz_A$. We have $W_{\mathbf{S}',i}(x) = -g'_i(x)$, $W_{\mathbf{A},i}(x) = -g'_i(x)x_A + z_A(xg'_i(x) - g_i(x))$ and $W_{\mathbf{B},i}(x) = -g'_i(x)x_B + z_B(xg'_i(x) - g_i(x))$ and so

$$\begin{aligned}\hat{R}_i(x) &= \begin{pmatrix} x \\ g_i(x) \\ 1 \end{pmatrix} \wedge \begin{pmatrix} -x_A(g'_i(x))^2x_B + z_Az_B(xg'_i(x) - g_i(x))^2 \\ 0 \\ -(g'_i(x))^2(xz_B + x_Bz_A) + 2z_Az_Bg'_i(x)(xg'_i(x) - g_i(x)) \end{pmatrix} \\ &= \begin{pmatrix} -g_i(x)(g'_i(x))^2(xz_B + x_Bz_A) + 2z_Az_Bg_i(x)g'_i(x)(xg'_i(x) - g_i(x)) \\ -x_A(g'_i(x))^2x_B + z_Az_B(xg'_i(x) - g_i(x))^2 - x[\dots] \\ x_Ag_i(x)(g'_i(x))^2x_B - z_Az_Bg_i(x)(xg'_i(x) - g_i(x))^2 \end{pmatrix},\end{aligned}$$

the valuation of the coordinates of which are larger than or equal to $2\text{val}(g_i) - 2$, the valuation of the second coordinate is $2\text{val}(g_i) - 2$. Hence $h_{m_1,i,\mathcal{B}} = 2\text{val}(g_i) - 2$ and the sum over $i = 1, \dots, e_{\mathcal{B}}$ of these quantities is equal to $2i_{m_1}(\mathcal{B}_0, \mathcal{T}_{m_1}\mathcal{B}_0) - 2e_{\mathcal{B}_0}$.

- Suppose that $I, J, S \in \mathcal{T}_{m_1}\mathcal{B}_0$ and $m_1 = J$. Take M such that $\mathbf{B}(0, 0, 1)$, $\mathbf{S}'(1, 0, 0)$, $y_A = 0$, $x_A \neq 0$ and $z_A \neq 0$. We have $W_{\mathbf{S}',i}(x) = -g'_i(x)$, $W_{\mathbf{A},i}(x) = -g'_i(x)x_A + z_A(xg'_i(x) - g_i(x))$ and $W_{\mathbf{B},i}(x) = xg'_i(x) - g_i(x)$ and so

$$\begin{aligned}\hat{R}_i(x) &= \begin{pmatrix} x \\ g_i(x) \\ 1 \end{pmatrix} \wedge \begin{pmatrix} z_A(xg'_i(x) - g_i(x))^2 \\ 0 \\ -x_A(g'_i(x))^2 + 2z_A(xg'_i(x) - g_i(x))g'_i(x) \end{pmatrix} \\ &= \begin{pmatrix} g_i(x)g'_i(x)(-g'_i(x)x_A + 2z_A(xg'_i(x) - g_i(x))) \\ z_A(xg'_i(x) - g_i(x))^2 - xg'_i(x)(-g'_i(x)x_A + 2z_A(xg'_i(x) - g_i(x))) \\ -g_i(x)z_A(xg'_i(x) - g_i(x))^2 \end{pmatrix},\end{aligned}$$

the valuation of the coordinates of which are larger than or equal to $2\text{val}(g_i) - 1$ and the valuation of the second coordinate is $2\text{val}(g_i) - 1$. Hence $h_{m_1,i,\mathcal{B}} = 2\text{val}(g_i) - 1$ and the sum over $i = 1, \dots, e_{\mathcal{B}}$ of these quantities is equal to $2i_{m_1}(\mathcal{B}_0, \mathcal{T}_{m_1}\mathcal{B}_0) - e_{\mathcal{B}_0}$.

- Suppose that $I, J, S \in \mathcal{T}_{m_1}\mathcal{B}_0$ and $m_1 = S$. Take M such that $\mathbf{S}'(0, 0, 1)$, $\mathbf{B}(1, 0, 0)$, $y_A = 0$, $x_A \neq 0$ and $z_A \neq 0$. We have $W_{\mathbf{B},i}(x) = -g'_i(x)$, $W_{\mathbf{A},i}(x) = -g'_i(x)x_A + z_A(xg'_i(x) - g_i(x))$ and $W_{\mathbf{S}',i}(x) = xg'_i(x) - g_i(x)$ and so

$$\begin{aligned} \hat{R}_i(x) &= \begin{pmatrix} x \\ g_i(x) \\ 1 \end{pmatrix} \wedge \begin{pmatrix} 2x_A g'_i(x)(xg'_i(x) - g_i(x)) - z_A(xg'_i(x) - g_i(x))^2 \\ 0 \\ (g'_i(x))^2 x_A \end{pmatrix} \\ &= \begin{pmatrix} g_i(x)(g'_i(x))^2 x_A \\ x_A g'_i(x)(xg'_i(x) - 2g_i(x)) - z_A(xg'_i(x) - g_i(x))^2 \\ -2x_A g_i(x)g'_i(x)(xg'_i(x) - g_i(x)) + z_A g_i(x)(xg'_i(x) - g_i(x))^2 \end{pmatrix}. \end{aligned}$$

The valuation of the first coordinate is $3\text{val}(g_i) - 2$ is smaller than or equal to the valuation of the third coordinate.

If $\text{val}(g_i) \neq 2$, the valuation of the second coordinate is $2\text{val}(g_i) - 1$; hence $h_{m_1,i,\mathcal{B}} = 2\text{val}(g_i) - 1$ and the sum over $i = 1, \dots, e_{\mathcal{B}}$ of these quantities is equal to $2i_{m_1}(\mathcal{B}_0, \mathcal{T}_{m_1}\mathcal{B}_0) - e_{\mathcal{B}_0}$.

Suppose now that $\text{val}(g_i) = 2$, then $3\text{val}(g_i) - 2 = 4$ and there exist $\alpha, \alpha_1 \in \mathbb{C}$ and $\beta_1 > 2$ such that $g_i(x) = \alpha x^2 + \alpha_1 x^{\beta_1} + \dots$. Then, the second coordinate has the following form $(x_A 2\alpha(\beta_1 - 2)x^{\beta_1+1} + \dots) + x^4(\dots)$. Therefore $h_{m_1,i,\mathcal{B}} = \min(\beta_1 + 1, 4)$ and the sum over $i = 1, \dots, e_{\mathcal{B}}$ of these quantities is equal to $e_{\mathcal{B}_0}(1 + \min(\beta_1, 3))$.

□

Proof of Theorem 10. Recall that (13) says

$$mclass(\Sigma_S(\mathcal{C})) = 2d^\vee + d - \sum_{m_1 \in \mathcal{C}} \sum_{\mathcal{B}_0 \in \text{Branch}_{m_1}(\mathcal{C})} h_{m_1, \mathcal{B}_0}$$

and that the values of h_{m_1, \mathcal{B}_0} have been given in Lemma 17.

- Assume first $S \notin \ell_\infty$. Then we have to sum the h_{m_1, \mathcal{B}_0} coming from Items 3, 4, 5, 6 and 7 of Lemma 17.

The sum of the h_{m_1, \mathcal{B}_0} coming from Items 3 and 5 applied with $m_1 = S$ gives directly g' .

The sum of the h_{m_1, \mathcal{B}_0} coming from Items 3, 5 and 7 applied with $m_1 \in \{I, J\}$ gives $f + \Omega_I(\mathcal{C}, \ell_\infty) + \Omega_J(\mathcal{C}, \ell_\infty)$.

The sum of the h_{m_1, \mathcal{B}_0} coming from Item 6 gives $g - \Omega_I(\mathcal{C}, \ell_\infty) - \Omega_J(\mathcal{C}, \ell_\infty)$.

The sum of the h_{m_1, \mathcal{B}_0} coming from Item 4 gives $2f' - q'$ (notice that $h_{m_1, \mathcal{B}_0} = 2(i_{m_1}(\mathcal{B}_0, \mathcal{T}_{m_1}\mathcal{B}_0) - e_{\mathcal{B}_0}) - (i_{m_1}(\mathcal{B}_0, \mathcal{T}_{m_1}\mathcal{B}_0) - 2e_{\mathcal{B}_0})\mathbf{1}_{i_{m_1}(\mathcal{B}_0, \mathcal{T}_{m_1}\mathcal{B}_0) \geq 2e_{\mathcal{B}_0}}$).

- Assume first $S \notin \ell_\infty$. Then we have to sum the h_{m_1, \mathcal{B}_0} coming from Items 3, 8, 9, 10 and 11 of Lemma 17.

The sum of the h_{m_1, \mathcal{B}_0} coming from Items 3 (with $m_1 = S$), 10 and 11 gives $2\Omega_S(\mathcal{C}, \ell_\infty) + \mu_S + c'(S)$.

The sum of the h_{m_1, \mathcal{B}_0} coming from Items 3 and 9 applied with $m_1 \in \{I, J\}$ gives $2(\Omega_I(\mathcal{C}, \ell_\infty) + \Omega_J(\mathcal{C}, \ell_\infty)) + \mu_I + \mu_J$.

The sum of the h_{m_1, \mathcal{B}_0} coming from Item 8 gives $2(g - \Omega_I(\mathcal{C}, \ell_\infty) - \Omega_J(\mathcal{C}, \ell_\infty) - \Omega_S(\mathcal{C}, \ell_\infty))$.

□

APPENDIX A. INTERSECTION NUMBERS OF CURVES AND PRO-BRANCHES

The following result expresses the classical intersection number $i_{m_1}(\mathcal{C}, \mathcal{C}')$ defined in [14, p. 54] thanks to the use of probranches.

Proposition 18. *Let $m \in \mathbb{P}^2$. Let $\mathcal{C} = V(F)$ and $\mathcal{C}' = V(F')$ be two algebraic plane curves containing m , with homogeneous polynomials $F, F' \in \mathbb{C}[X, Y, Z]$. Let $M \in GL(\mathbb{C}^3)$ be such that $\Pi(M(\mathbf{P}_0)) = m$ and such that the tangent cones of $V(F \circ M)$ and of $V(F' \circ M)$ do not contain $X = 0$.*

Assume that $V(F \circ M)$ admits b branches at P_0 and that its β -th branch \mathcal{B}_β has multiplicity e_β . Assume that $V(F' \circ M)$ admits b' branches at P_0 and that its β' -th branch $\mathcal{B}'_{\beta'}$ has multiplicity $e'_{\beta'}$.

Then we have

$$i_m(\mathcal{C}, \mathcal{C}') = \sum_{\beta=1}^b \sum_{j=0}^{e_\beta-1} \sum_{\beta'=1}^{b'} \sum_{j'=0}^{e'_{\beta'}-1} \text{val}_x[h_\beta(\zeta^j x^{\frac{1}{e_\beta}}) - h'_{\beta'}(\zeta'^{j'} x^{\frac{1}{e'_{\beta'}}})],$$

with $y = h_\beta(\zeta^j x^{\frac{1}{e_\beta}}) \in \mathbb{C}\langle x^ \rangle$ an equation of the j -th probranch of \mathcal{B}_β at P_0 , $y = h'_{\beta'}(\zeta'^{j'} x^{\frac{1}{e'_{\beta'}}}) \in \mathbb{C}\langle x^* \rangle$ an equation of the j' -th probranch of $\mathcal{B}'_{\beta'}$ at P_0 , with $\zeta := e^{\frac{2i\pi}{e_\beta}}$ and $\zeta' := e^{\frac{2i\pi}{e'_{\beta'}}}$.*

With the notations of Proposition 18, we get

$$i_m(\mathcal{C}, \mathcal{C}') = \sum_{\beta=1}^b i_{P_0}(\mathcal{B}_\beta, V(F')), \quad (14)$$

with the usual definition given in [21] of intersection number of a branch with a curve

$$i_{P_0}(\mathcal{B}_\beta, V(F' \circ M)) = \sum_{j=0}^{e_\beta-1} \text{val}_x(F' \circ M(x, h_{j,\beta}(\zeta^j x^{\frac{1}{e_\beta}}))). \quad (15)$$

Proof of Proposition 18. By definition, the intersection number is defined by

$$i_m(\mathcal{C}, \mathcal{C}') = i_{P_0}(V(F \circ M, F' \circ M)) = \text{length} \left(\left(\frac{\mathbb{C}[X, Y, Z]}{(F \circ M, F' \circ M)} \right)_{(X, Y, Z)} \right)$$

where $(\frac{\mathbb{C}[X, Y, Z]}{(F \circ M, F' \circ M)})_{(X, Y, Z)}$ is the local ring in the maximal ideal (X, Y, Z) of P_0 [14, p. 53]. According to [12], we have

$$i_m(\mathcal{C}, \mathcal{C}') = \dim_{\mathbb{C}} \left(\left(\frac{\mathbb{C}[X, Y, Z]}{(F \circ M, F' \circ M)} \right)_{(X, Y, Z)} \right)$$

Let f, f' be defined by $f(x, y) = F \circ M(x, y, 1)$, $f'(x, y) = F' \circ M(x, y, 1)$. We get

$$i_m(\mathcal{C}, \mathcal{C}') = \dim_{\mathbb{C}} \left(\left(\frac{\mathbb{C}[x, y]}{(f, f')} \right)_{(x, y)} \right) = \dim_{\mathbb{C}} \frac{\mathbb{C}\langle x, y \rangle}{(f, f')}.$$

Recall that, according to the Weierstrass preparation theorem, there exist two units U and U' of $\mathbb{C}\langle x, y \rangle$ and $f_1, \dots, f_b, f'_1, \dots, f'_{b'} \in \mathbb{C}\langle x \rangle[y]$ monic irreducible such that

$$f = U \prod_{\beta=1}^b f_\beta \quad \text{and} \quad f' = U' \prod_{\beta'=1}^{b'} f'_{\beta'},$$

$f_\beta = 0$ being an equation of \mathcal{B}_β and $f'_{\beta'} = 0$ being an equation of $\mathcal{B}'_{\beta'}$. According to the Puiseux theorem, \mathcal{B}_β (resp. $\mathcal{B}'_{\beta'}$) admits a parametrization

$$\left\{ \begin{array}{l} x = t^{e_\beta} \\ x = h_\beta(t) \in \mathbb{C}\langle t \rangle \end{array} \right. \quad (\text{resp.} \quad \left\{ \begin{array}{l} x = t^{e'_{\beta'}} \\ x = h'_{\beta'}(t) \in \mathbb{C}\langle t \rangle \end{array} \right.).$$

We know that, for every $\beta \in \{1, \dots, b\}$ and every $j \in \{0, \dots, e_\beta\}$, $h_\beta(\zeta^j x^{\frac{1}{e_\beta}}) \in \mathbb{C}\langle x^{\frac{1}{e_\beta}} \rangle$ are the y -roots of f_β (resp. $h_{\beta'}(\zeta'^j x^{\frac{1}{e'_{\beta'}}}) \in \mathbb{C}\langle x^{\frac{1}{e'_{\beta'}}} \rangle$ are the y -roots of $f'_{\beta'}$). In particular, we have

$$f_\beta(x, y) = \prod_{j=0}^{e_\beta-1} (y - h_\beta(\zeta^j x^{\frac{1}{e_\beta}})) \quad \text{and} \quad f'_{\beta'}(x, y) = \prod_{j'=0}^{e'_{\beta'}-1} (y - h'_{\beta'}(\zeta'^{j'} x^{\frac{1}{e'_{\beta'}}})).$$

Therefore we have the following sequence of \mathbb{C} -algebra-isomorphisms:

$$\frac{\mathbb{C}\langle x, y \rangle}{(f, f')} = \frac{\mathbb{C}\langle x, y \rangle}{(\prod_{\beta=1}^b f_\beta(x, y), f'(x, y))} \cong \prod_{\beta=1}^b A_\beta,$$

where $A_\beta := \frac{\mathbb{C}\langle x, y \rangle}{(f_\beta(x, y), f'(x, y))}$. Let $\beta \in \{1, \dots, b\}$. We observe that we have

$$A_\beta = \prod_{j=0}^{e_\beta-1} \frac{\mathbb{C}\langle x \rangle}{(f'(x, h_\beta(\zeta^j x^{\frac{1}{e_\beta}})))}.$$

On another hand, we have

$$\begin{aligned} D_\beta &:= \frac{\mathbb{C}\langle x^{\frac{1}{e_\beta}}, y \rangle}{(f_\beta(x, y), f'(x, y))} = \frac{\mathbb{C}\langle x^{\frac{1}{e_\beta}}, y \rangle}{(\prod_{j=0}^{e_\beta-1} (y - h_\beta(\zeta^j x^{\frac{1}{e_\beta}})), f'(x, y))} \\ &\cong \prod_{j=0}^{e_\beta-1} \frac{\mathbb{C}\langle x^{\frac{1}{e_\beta}}, y \rangle}{(y - h_\beta(\zeta^j x^{\frac{1}{e_\beta}}), f'(x, y))} \cong \prod_{j=0}^{e_\beta-1} D_{\beta,j} \end{aligned}$$

with

$$D_{\beta,j} := \frac{\mathbb{C}\langle x^{\frac{1}{e_\beta}} \rangle}{(f'(x, h_\beta(\zeta^j x^{\frac{1}{e_\beta}})))}.$$

We consider now the natural extension of rings $i_\beta : A_{\beta,j} \hookrightarrow D_{\beta,j}$ such that

$$\forall g \in A_\beta, \quad \text{val}_{x^{1/e_\beta}}((i_\beta(g))(x)) = e_\beta \text{val}_x(g(x)).$$

We have

$$D_\beta \cong \prod_{j=0}^{e_\beta-1} \frac{\mathbb{C}\langle x^{\frac{1}{e_\beta}} \rangle}{(x^{v_\beta})},$$

where v_β is the valuation in $x^{\frac{1}{e_\beta}}$ of $(f'(x, h_\beta(\zeta^j x^{\frac{1}{e_\beta}})))$, i.e.

$$v_\beta := \text{val}_t(f'(t^{e_\beta}, h_\beta(\zeta^j t))) = e_\beta \text{val}_x(f'(x, h_\beta(\zeta^j x^{\frac{1}{e_\beta}}))).$$

We get

$$\begin{aligned} i_m(\mathcal{C}, \mathcal{C}') &= \sum_{\beta=1}^b \dim_{\mathbb{C}} A_{\beta} = \sum_{\beta=1}^b \sum_{j=0}^{e_{\beta}-1} \frac{1}{e_{\beta}} \text{val}_t(f'(t^{e_{\beta}}, h_{\beta}(\zeta^j t))) \\ &= \sum_{\beta=1}^b \sum_{j=0}^{e_{\beta}-1} \text{val}_x(f'(x, h_{\beta}(\zeta^j x^{\frac{1}{e_{\beta}}}}))) = \sum_{\beta=1}^b \sum_{j=0}^{e_{\beta}-1} \sum_{\beta'=1}^{b'} \text{val}_x(f'_{\beta'}(x, h_{\beta}(\zeta^j x^{\frac{1}{e_{\beta}}}}))). \end{aligned}$$

Observe now that

$$\text{val}_x(f'_{\beta'}(x, h_{\beta}(\zeta^j x^{\frac{1}{e_{\beta}}}}))) \in \frac{1}{e_{\beta}} \mathbb{N}$$

and that

$$f'_{\beta'}(x, h_{\beta}(\zeta^j x^{\frac{1}{e_{\beta}}})) \equiv \text{Res}(f'_{\beta'}, f_{\beta}; y) \equiv \prod_{j'=0}^{e'_{\beta'}-1} (h'_{\beta'}(\zeta'^{j'} x^{\frac{1}{e'_{\beta'}}}) - h_{\beta}(\zeta^j x^{\frac{1}{e_{\beta}}})) ,$$

where Res denotes the resultant and where \equiv means "up to a non zero scalar". Finally, we get

$$i_m(\mathcal{C}, \mathcal{C}') = \sum_{\beta=1}^b \sum_{j=0}^{e_{\beta}-1} \sum_{\beta'=1}^{b'} \sum_{j'=0}^{e'_{\beta'}-1} \text{val}_x[h'_{\beta'}(\zeta'^{j'} x^{\frac{1}{e'_{\beta'}}}) - h_{\beta}(\zeta^j x^{\frac{1}{e_{\beta}}})].$$

□

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